7.5.4

a) $\ddot{x} + \mu f(x)\dot{x} + x = 0$ can be ~~recolo~~ recast into ~~the~~ a first order system.

~~[illegible]~~ $\ddot{x} + \mu f(x)\dot{x} = \frac{d}{dt}\left[\dot{x} + \mu \int^{x} f(x')\,dx'\right]$

Since $f(x) = \begin{cases} -1 & , |x| < 1 \\ 1 & , |x| > 1 \end{cases}$

$\Rightarrow F(x) = \int^{x} f(x')\,dx' = \begin{cases} x + C_1 & , x \leq -1 \\ -x + C_2 & , |x| < 1 \\ x + C_3 & , x \geq 1 \end{cases}$

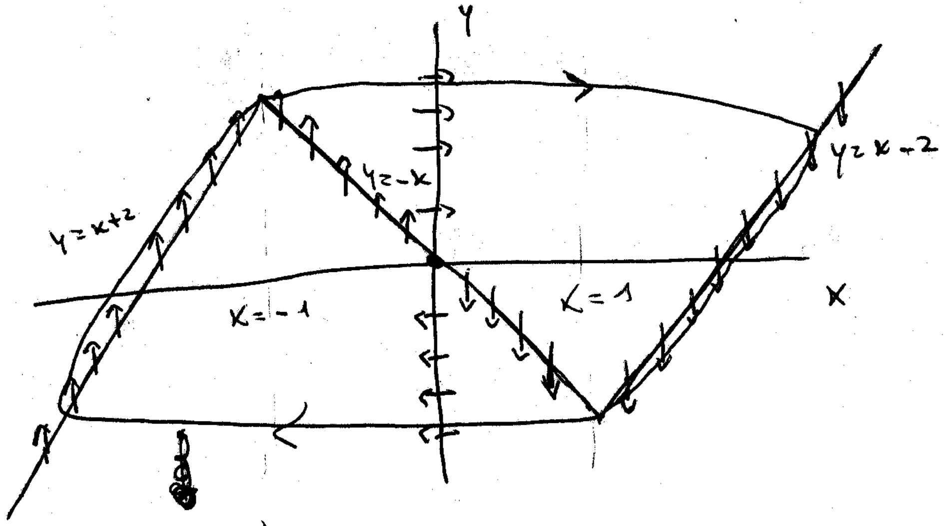
by choosing $C_1 = 2$, $C_2 = 0$, $C_3 = -2$ we ~~have~~ ensure the continuity of $F(x)$

So now $\frac{d}{dt}\left[\dot{x} + \mu F(x)\right] = -x$ and setting

$\mu y = \mu F(x) + \dot{x}$ we get

$$\begin{cases} \dot{x} = \mu\left[y - F(x)\right] \\ \dot{y} = -\frac{x}{\mu} \end{cases}$$

b)



c) exact same conjecture as in example 7.5.1 of Strogatz's.

d) $T \cong 2 \int_{t_a}^{t_b} dt$ along the slow branches

let's consider the "positive" branch ($x \geq 1$, $F(x) = x - 2$)

then $\frac{dy}{dt} = F'(x) \frac{dx}{dt} = \frac{dx}{dt}$

On the other hand, $\frac{dy}{dt} = -\frac{x}{\mu}$

$\Rightarrow dt = -\frac{dx}{x} \mu$

$\Rightarrow T = 2 \int_3^1 \left(-\frac{\mu dx}{x}\right) = 2\mu \ln 3$

7.6.3

a) $\ddot{x} + x = \epsilon$

a particular solution to the problem is $x = \epsilon$

the general solution to the homogeneous equation is

$$x = A\cos t + B\sin t$$

$$\Rightarrow \quad x(t) = A\cos t + B\sin t + \epsilon$$

$$x(0) = 1 = A + \epsilon \Rightarrow A = 1 - \epsilon$$

$$\dot{x}(0) = 0 = B \Rightarrow B = 0$$

b) $\ddot{x}_0 + \epsilon\ddot{x}_1 + \epsilon^2\ddot{x}_2 + x_0 + \epsilon x_1 + \epsilon^2 x_2 = \epsilon$

i) $\ddot{x}_0 + x_0 = 0 \Rightarrow x_0 = C_1\cos t + C_2\sin t$

$x_0(0) = 1, \ \dot{x}_0(0) = 0 \Rightarrow x_0 = \cos t$

ii) $\ddot{x}_1 + x_1 = 1 \Rightarrow x_1 = C_3\cos t + C_4\sin t + 1$

$\dot{x}_1(0) = x_1(0) = 0 \Rightarrow C_3 = -1, \ C_4 = 0$

iii) $\ddot{x}_2 + x_2 = 0 \Rightarrow x_2 = C_5\cos t + C_6\sin t$

$\dot{x}_2(0) = x_2(0) = 0 \Rightarrow C_5 = C_6 = 0$

$$\Rightarrow x(t) = \cos t + \epsilon\left[1 - \cos t - \sin t\right] =$$

$$(1-\epsilon)\cos t + \epsilon$$

c) There are no secular terms or resonant forcings since the driving term ϵ isn't coupled to the variable x, which makes all the differential equations in b) independent of one another. Also ϵ isn't coupled to any time-dependent term, so that those equations are also not explicitly time-dependent. That rules out the presence of any resonance.

7.6.6. To leading order we have

$$x = r(T)\cos(\tau + \varphi(T)) + O(\varepsilon)$$

$$\Rightarrow \dot{x} = \frac{d}{dt}x = \frac{d}{d\tau}x + O(\varepsilon) = -r(T)\sin(\tau + \varphi(T)) \quad \Big\} \ \tau + \varphi = \theta$$

$$\Rightarrow h(x, x') = -r^2 \sin\theta\cos\theta$$

Using averaging theory,

$$\partial_T r = \langle h \sin\theta \rangle = \frac{1}{2\pi}\int_0^{2\pi} (-r^2)\sin^2\theta\cos\theta = 0 \quad \Rightarrow r = r_0$$

$$\partial_T \varphi = \frac{1}{r}\langle h\cos\theta\rangle = \frac{1}{2\pi}\int_0^{2\pi} (-r)\sin\theta\cos^2\theta = 0 \quad \Rightarrow \varphi = \varphi_0$$

with some constants φ_0 and r_0 such that

$$\begin{cases} x(0) = r_0\cos(\varphi_0) = a \\ \dot{x}(0) = -r_0\sin(\varphi_0) = 0 \end{cases} \Rightarrow \begin{bmatrix} \varphi_0 = 0 \\ r_0 = a \end{bmatrix}$$

Putting everything together, we find

$$x(t, \varepsilon) = a\cos t + O(\varepsilon),$$

that is the nonlinearity has <u>no effect</u> on the solution at zero-th order in perturbation theory. There will be an effect at higher orders, however.

The result, to leading order in ε, suggests that there are no limit cycles; instead the oscillator possesses a continuum of closed orbits, all with period 2π, or unit frequency.

7.6.17

a) $\ddot{x} + \left(1 + \epsilon\gamma + \epsilon\cos 2t\right)x = 0$

$$h = \epsilon\gamma x + \epsilon\cos 2t\, x = \gamma\epsilon r\cos\theta + \epsilon r\cos^2\theta\cos 2t$$

so that

$$r' = \frac{1}{2\pi}\int_0^{2\pi}\gamma\epsilon r\cos\theta\sin\theta + \int_0^{2\pi} r\cos 2t\cos\theta\sin\theta =$$

$$\frac{1}{4} r\sin 2\phi$$

$$r\phi' = \frac{1}{2\pi}\int_0^{2\pi}\gamma r\cos^2\theta\, d\theta + \int_0^{2\pi} r\cos 2t\cos^2\theta\, d\theta =$$

$$\frac{1}{2} r\left(\gamma + \frac{1}{2}\cos 2\phi\right)$$

b)-c) If $\phi'' \gg r'$ we can assume stationary state for ϕ, i.e. $\phi' = 0 \Rightarrow \gamma = -\frac{1}{2}\cos 2\phi$

c) $\sin 2\phi = \sqrt{1-4\gamma^2}$ real only if

$$|\gamma| < 1/2$$

In that case $r' = \frac{1}{4} r \sqrt{1-4\gamma^2}$,

so that it increases exponentially w/ rate $k = \frac{1}{4}\sqrt{1-4\gamma^2}$

d) the best way to see this is probably to consider

$$\frac{d\phi}{dT} = \frac{1}{2}\left(\gamma + \frac{1}{2}\cos 2\phi\right)$$

$$\Rightarrow \quad T = \int \frac{d(2\phi)}{\gamma + \frac{1}{2}\cos 2\phi} = \frac{2}{\sqrt{\gamma^2 - 1/4}} \operatorname{arctg}\left[\sqrt{\frac{\gamma - 1/2}{\gamma + 1/2}}\, \operatorname{tg}\phi\right]$$

if $|\gamma| > 1/2$, so that

$$\sqrt{\frac{\gamma - 1/2}{\gamma + 1/2}}\, \operatorname{tg}\phi = \operatorname{tg}\left[\frac{\sqrt{\gamma^2 - 1/4}}{2}\, T\right]$$

that implies $\phi \propto T$ and

$r(T) \propto e^{\cos aT}$ or something that oscillates

anyway, so that $x = r(T)\cos\left[t + \phi(T)\right]$

also does oscillate.

e) γ is related to ~~the~~ the gravity, so that,

from the results, it is possible to figure what it means:

if $|\gamma| < \frac{1}{2}$ the ~~oscillations grow exponentially~~

~~there are no~~ oscillations grow exponentially,

meaning too little gravity to balance the ~~[illegible]~~ periodic

swing. If $|\gamma| > \frac{1}{2}$, then the oscillations are

under control, meaning the gravitational parameter is

big enough.