8.1.8

HW #10

the corresponding system to the ~~equat~~ given equation reads

$$\phi' = y$$

$$y' = -\frac{y}{\sigma} - \frac{1}{\sigma}\sin\phi + \frac{\gamma}{\sigma}\sin\phi\cos\phi \quad , \quad \gamma, \sigma > 0$$

Fixed points of this system occur at $(0,0)$, $(\pi, 0)$ when $\gamma < 1$ and at $(0,0)$, $(\pi, 0)$, $(\arccos\left(\frac{1}{\gamma}\right), 0)$ → which makes two points.

the Jacobian reads

$$J = \left(\begin{array}{c|c} 0 & 1 \\ \hline -\frac{1}{\sigma}\cos\phi + \frac{\gamma}{\sigma}\cos 2\phi & -1/\sigma \end{array}\right)$$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{\sigma} + \frac{\gamma}{\sigma} & -1/\sigma \end{pmatrix}$$

so that the eigenvalues read

$$\lambda = \frac{-\frac{1}{\sigma} \pm \sqrt{\frac{1}{\sigma^2} - \frac{4}{\sigma} + \frac{4\gamma}{\sigma}}}{2}$$

The condition for one of the eigenvalues to be positive is

$$\frac{1}{\sigma} < \sqrt{\frac{1}{\sigma^2} - \frac{4}{\sigma} + \frac{4\gamma}{\sigma}}$$

$$\Rightarrow \quad \gamma > 1$$

~~[illegible]~~ the condition for the eigenvalues to be real is

$$\frac{1}{\sigma^2} - \frac{4}{\sigma} + \frac{4\gamma}{\sigma} > 0 \iff \gamma > \frac{4\sigma - 1}{4\sigma}$$

Let's now consider the point $(\pi, 0)$

$$J(\pi, 0) = \begin{pmatrix} 0 & 1 \\ \frac{1}{\sigma} + \frac{\gamma}{\sigma} & -\frac{1}{\sigma} \end{pmatrix}$$

$$\Rightarrow \quad \lambda = -\frac{1}{2\sigma} \pm \frac{1}{2}\sqrt{\frac{1}{\sigma^2} + \frac{4}{\sigma} + \frac{4\gamma}{\sigma}}$$

which always means ~~one~~ positive and one negative eigenvalues, i.e. a saddle.

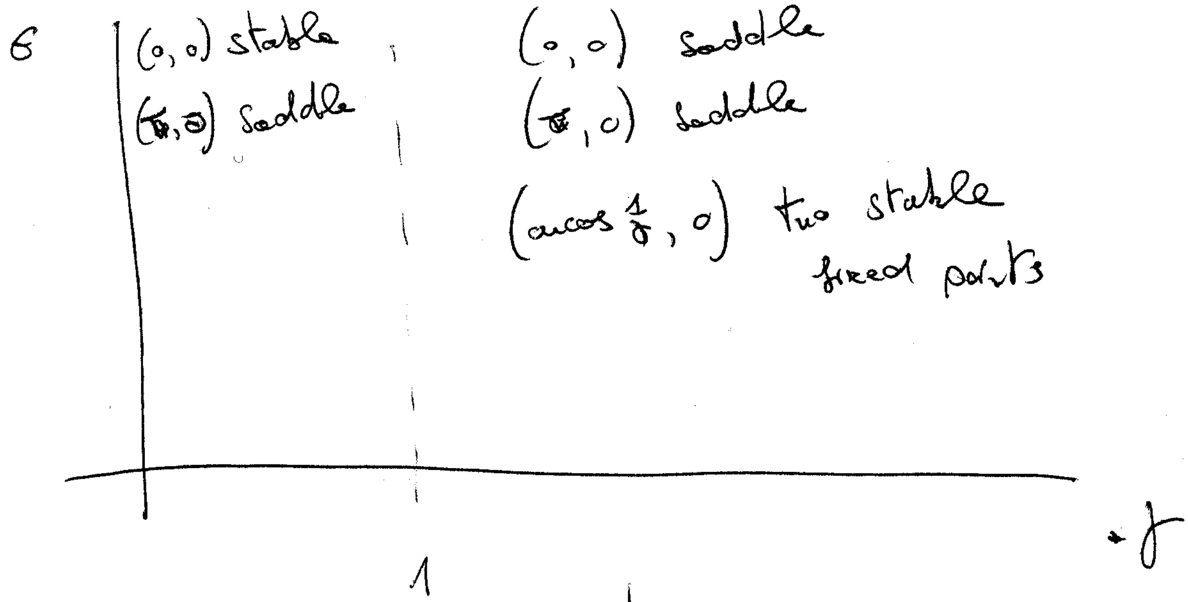
Let's now look at the points that verify $\cos\phi = \frac{1}{\gamma}$

$$J = \begin{pmatrix} 0 & 1 \\ -\frac{\gamma}{\sigma} + \frac{1}{\gamma\sigma} & -1/\sigma \end{pmatrix}$$

$$\Rightarrow \lambda = \frac{-\frac{1}{\sigma} \pm \sqrt{\frac{1}{\sigma^2} + \frac{4}{\gamma\sigma} - \frac{4\gamma}{\sigma}}}{2}$$

the condition for having a positive eigenvalues turns out to be $\gamma < 1$, in which case, however, these two fixed points do not exist. Therefore they are always stable.

b)



pitchfork bifurcation occurring at $\gamma = 1$

8.1.11

$$\begin{cases} \dot{u} = a(1-u) - uv^2 \\ \dot{v} = uv^2 - (a+k)v \end{cases} \quad , a, k > 0$$

the condition for fixed points reads

$$0 = a(1-u) - uv^2$$

$$0 = uv^2 - (a+k)v$$

$$\Leftrightarrow \quad a(1-u) = (a+k)v \quad \rightarrow \quad v = \frac{a(1-u)}{(a+k)}$$

plug it into the first equation and get

$$0 = a(1-u) - u\frac{a^2(1-u)^2}{(a+k)^2}$$

$$\Leftrightarrow 1 - \frac{ua(1-u)}{(a+k)^2} = 0 \Leftrightarrow (a+k)^2 - au + au^2 = 0$$

$$\Leftrightarrow \quad u = \frac{a \pm \sqrt{a^2 - 4a(a+k)^2}}{2a}$$

the condition for the point to exist is

$$a - 4(a+k)^2 > 0 \quad \Leftrightarrow$$

$$a - 4a^2 - 4k^2 - 8ak > 0 \quad \Leftrightarrow$$

$$-a - \frac{1}{2}\sqrt{a} < k < -a + \frac{1}{2}\sqrt{a}$$

8.2.1 $\ddot{x} + \mu(x^2-1)\dot{x} + x = a$

$$\Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = -\mu(x^2-1)y - x + a \end{cases}$$

The only fixed point is ~~occurred~~ $(a,0)$

$$J(a,0) = \begin{pmatrix} 0 & 1 \\ -1 & -\mu(a^2-1) \end{pmatrix}$$

of eigenvalues

$$\lambda = \frac{-\mu(a^2-1) \pm \sqrt{\mu^2(a^2-1)^2 - 4\mu(a^2-1)}}{2}$$

so that, provided that $\mu^2(a^2-1)^2 < 4\mu(a^2-1)$

(two complex conjugate eigenvalues)

the real part of changes sign when

$\mu(a^2-1)$ changes sign, that is

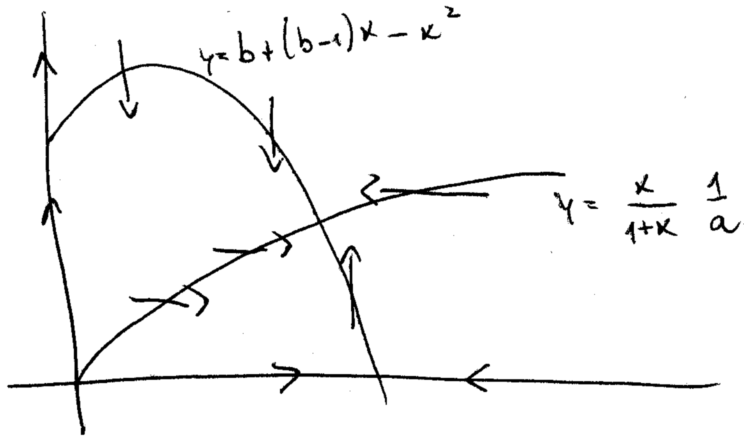
the curves we're looking for are $\begin{cases} \mu = 0 \\ a = \pm 1 \end{cases}$

8.2.9

a)-b)
$$\begin{cases} \dot{x} = x\left(b - x - \dfrac{y}{1+x}\right) \\ \dot{y} = y\left(\dfrac{x}{1+x} - ay\right) \end{cases}$$

Nullclines are $x = 0$, $y = 0$

$$y = \frac{x}{1+x}\frac{1}{a}, \quad y = b + (b-1)x - x^2$$



there are always three intersections between nullclines, therefore no tangent or pitchfork bifurcations may occur.

(Recall $x, y \geq 0$, $a, b > 0$)

c) $J = \begin{pmatrix} b - 2x - \frac{y}{(1+x)^2} & -\frac{x}{1+x} \\ \frac{y}{(1+x)^2} & \frac{x}{1+x} - 2ay \end{pmatrix}$

$$z(x^*, y^*) = b - 2x^* - \frac{y^*}{(1+x^*)^2} - 2ay^* + \frac{x^*}{1+x^*}$$

Let me replace the first y with $(b-x^*)(1+x^*)$ (first nullcline)
and the second with $\frac{1}{a}\frac{x^*}{1+x^*}$ (second nullcline)

So that

$$z = b - 2x^* - \frac{b-x^*}{1+x^*} + \frac{x^*}{1+x^*} - \frac{2x^*}{1+x^*}$$

$$= b - 2x^* - \frac{b-x^*}{1+x^*} - \frac{x^*}{1+x^*} = 0$$

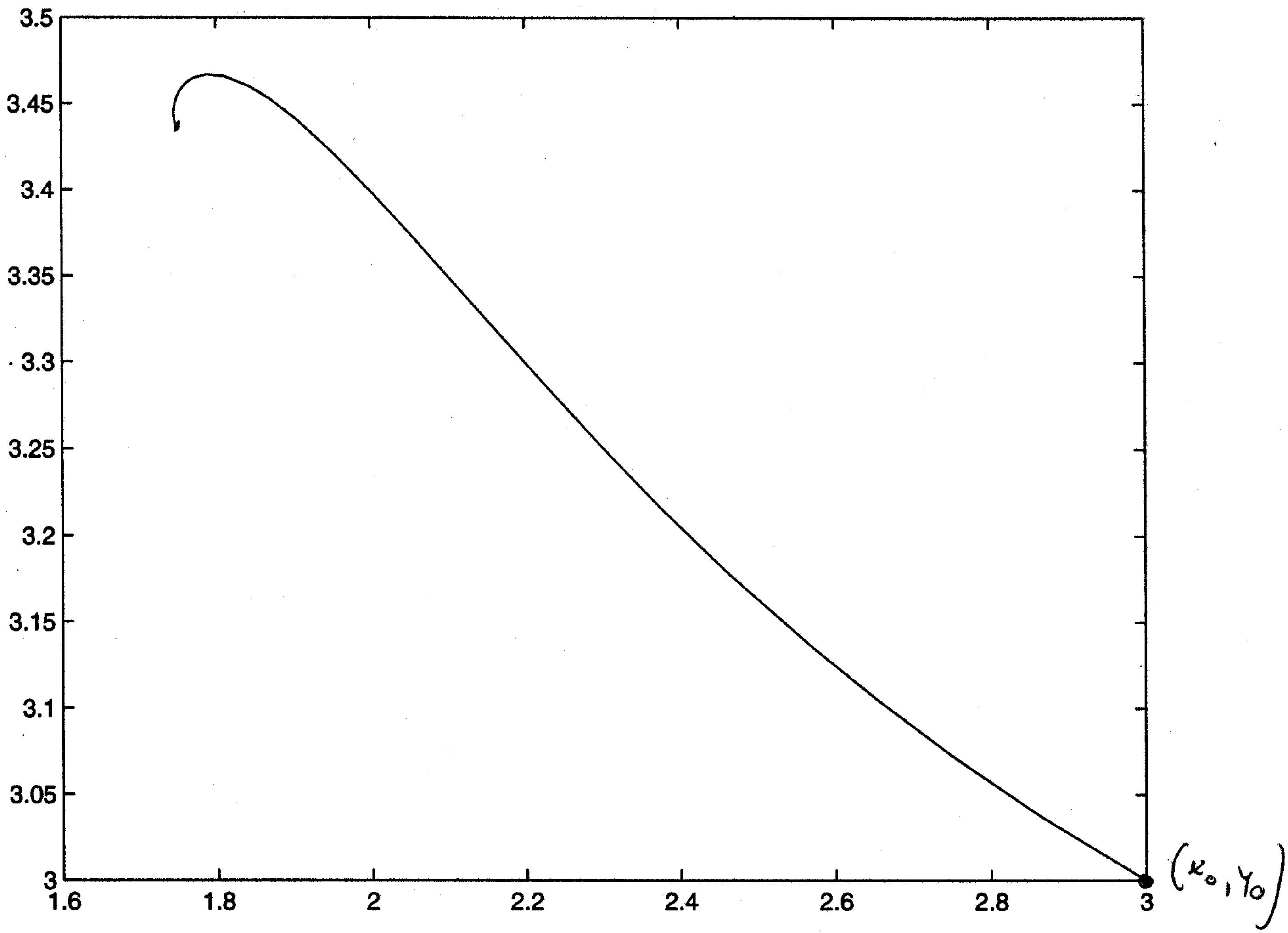
$$\Leftrightarrow \quad x^* = \frac{b-2}{2}$$

Now $\begin{cases} y^* = \frac{1}{a}\frac{x^*}{1+x^*} \\ y^* = (b-x^*)(1+x^*) \end{cases} \Rightarrow (b-x^*)(1+x^*) = \frac{1}{a}\frac{x^*}{1+x^*}$

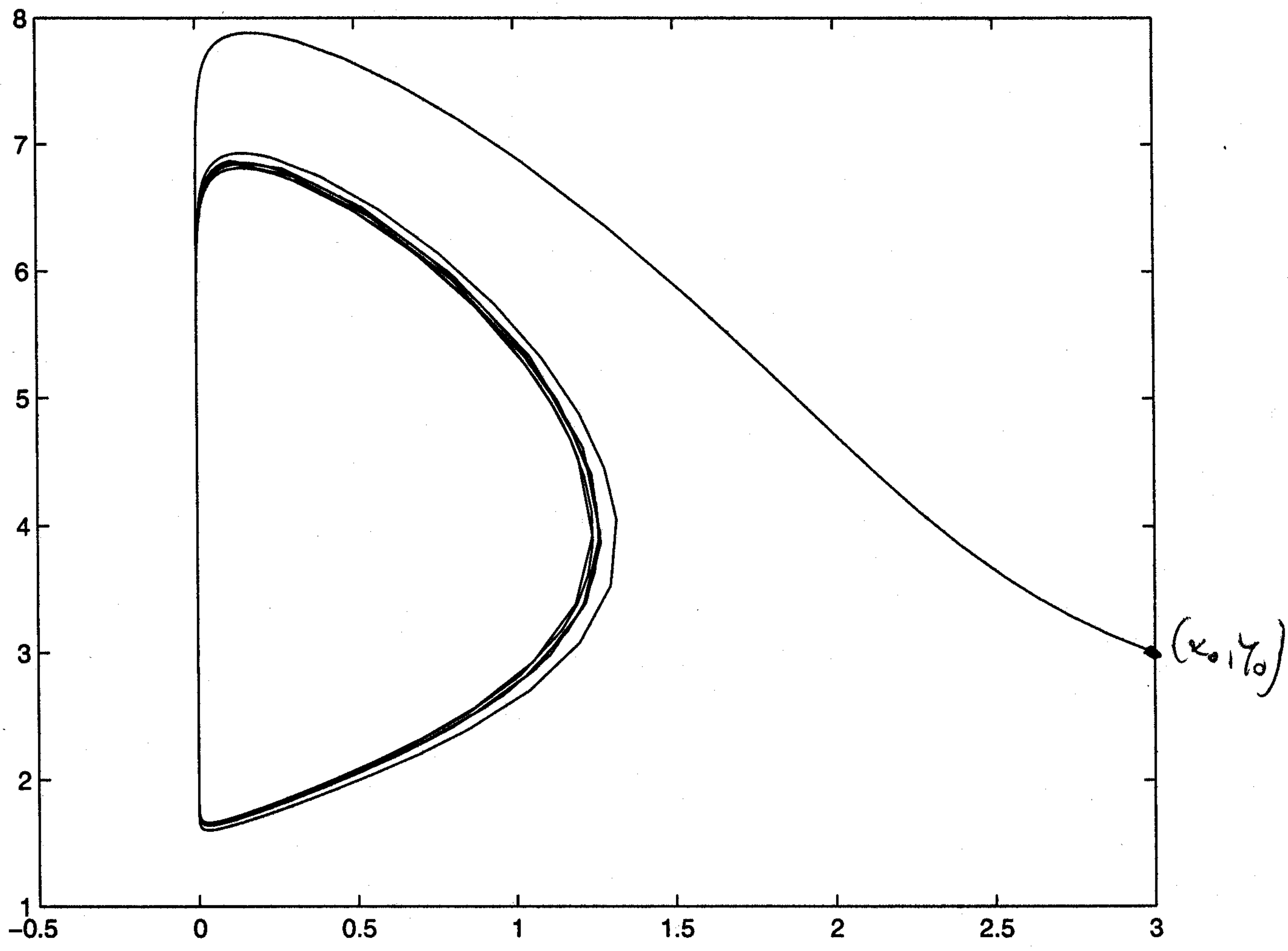
but $x^* = \frac{b-2}{2}$, so that $\left(b - \frac{b-2}{2}\right)\left(1 + \frac{b-2}{2}\right) = \frac{1}{a}\frac{\frac{b-2}{2}}{1+\frac{b-2}{2}}$

$$\Leftrightarrow \quad a = \frac{4(b-2)}{b^2(b+2)}$$

d) Example of $b > b_c$, the initial condition ~~ends in the~~ ends in the fixed point (x^*, y^*)



Here $b < b_c$, the same initial condition converges to a stable cycle, i.e. the bifurcation is supercritical.



8.3.1.

a)

$$\dot{x} = 1-(b+1)x + ax^2y$$

$$\dot{y} = bx - ax^2y$$

$a, b > 0$, $x, y \geq 0$

$$bx - ax^2y = 0 \iff \begin{cases} x = 0 & \text{or} \\ b - axy = 0 \iff y = \dfrac{b}{ax} \end{cases}$$

$$\dot{x} = 1-(b+1)x + bx = 1 - x = 0$$

$$\iff x = 1,\ y = \frac{b}{a} \quad \text{only fixed point}$$

$$J = \begin{pmatrix} -(b+1) + 2axy & ax^2 \\ b - 2axy & -ax^2 \end{pmatrix}$$

$$J\left(1, \frac{b}{a}\right) = \begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix}$$

$$(b-1-\lambda)(-a-\lambda)+ab=0 \iff$$

$$-\cancel{a}b-\lambda b+a+\lambda+\lambda a+\lambda^2+\cancel{a}b=0$$

$$\iff \lambda = \frac{b-a-1 \pm \sqrt{b^2+a^2+1-2b+2a-2ab-4a}}{2}$$

$$= \frac{b-a-1 \pm \sqrt{b^2+a^2+1-2b-2a-2ab}}{2}$$

if $b^2+a^2+1-2b-2a-2ab<0$

then we have either a stable or an unstable spiral, depending on the sign of $b-a-1$.

if $b^2+a^2+1-2b-2a-2ab \geq 0$, then

$$(b-a-1) > \sqrt{b^2+a^2+1-2b-2a-2ab} \iff$$

$$\cancel{b^2}+\cancel{a^2}+\cancel{1}-2\cancel{a}b-\cancel{2b}+2a > \cancel{b^2}+\cancel{a^2}+\cancel{1}-\cancel{2b}-2a-2\cancel{a}b$$

$$\iff 4a>0 \quad \forall a,$$

meaning the point is either a stable or an unstable node, depending on the sign of $b-a-1$, but never a saddle.

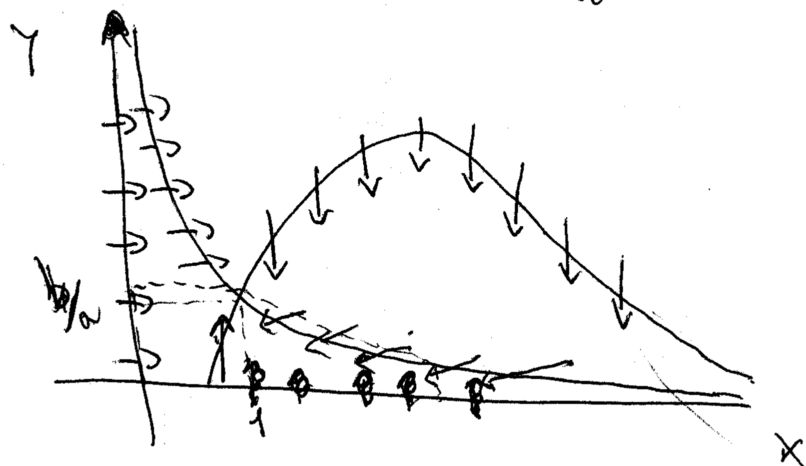
b)

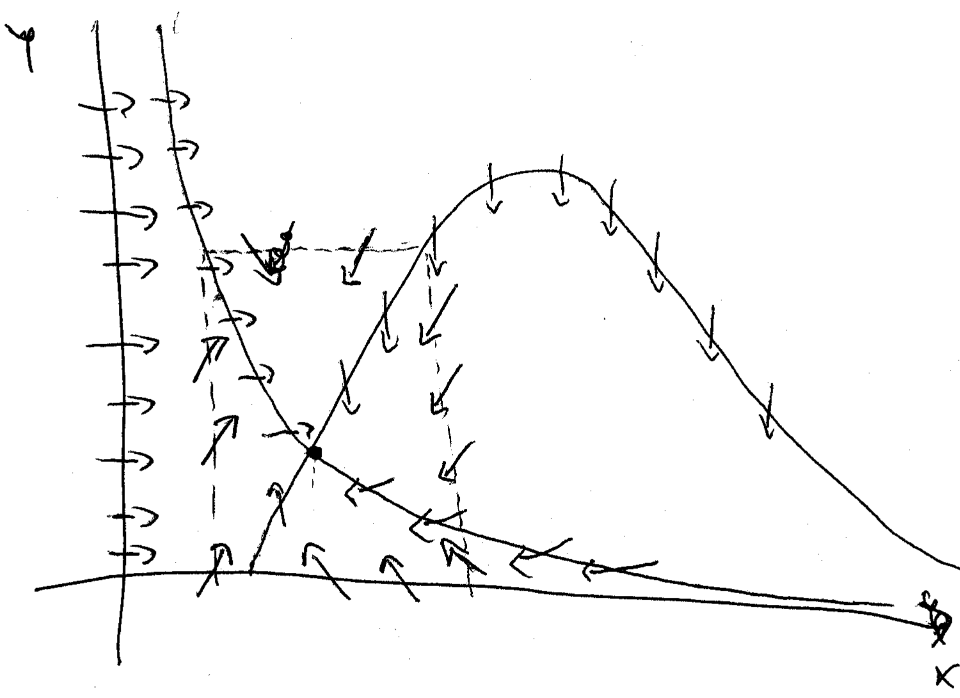
i) $1-(b+1)x+ax^2y=0$

$\iff y=\dfrac{(b+1)x-1}{ax^2}$

ii) $bx-ax^2y=0 \iff y=\dfrac{b}{ax}$

iii) $x=0$





c) if $\sqrt{b^2+a^2+1-2b-2a-2ab}$ is imaginary, then a Hopf bifurcation occurs ~~also~~ at $b = a+1$

d) if $b > b_c$, ~~then~~ we have an unstable spiral, however the flow can't get out of the trapping region sketched in b), therefore there must exist a limit cycle.

e) $T = \dfrac{2\pi}{\omega}$, $\omega = a$ (cf. Strogatz page 260 and ~~see~~ question a) of this problem)

$\Rightarrow T \simeq \dfrac{2\pi}{a}$

for $b \approx b_c$