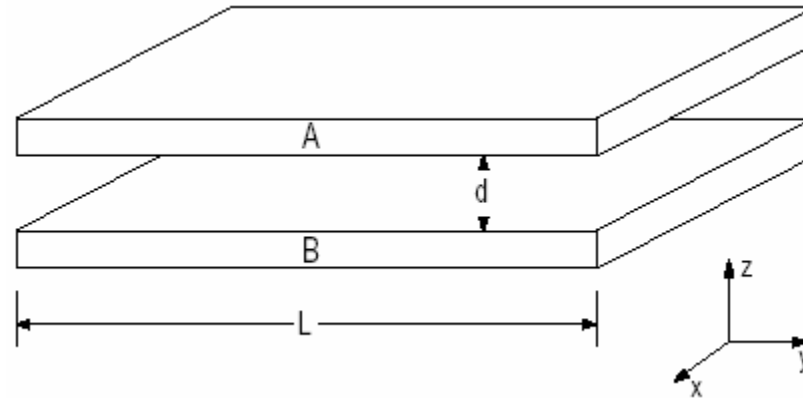


In the following we will (almost always) assume translational invariance, along with the other idealizations to facilitate the analytical approach. Initially we will also restrict the stability analysis to uniform (featureless) states, which are explicitly translationally invariant.

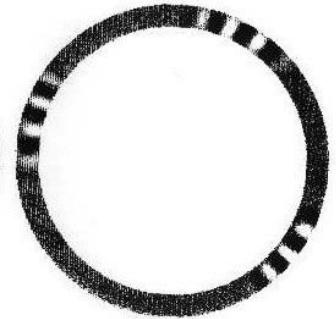


An idealized pattern-forming system consisting of a continuous medium between two flat spatially-uniform parallel plates A and B that are separated by a distance  $d$ . The medium is driven out of equilibrium by gradients of energy, momentum, or matter that cause these quantities to flow from one plate to the other. Because the plates are uniform, the gradients are normal to the plates. Coordinates parallel to the gradients (here  $z$ ) are denoted by the vector  $\mathbf{x}_{\parallel}$  and are called confined. Coordinates perpendicular to the gradients (here  $x$  and  $y$ ) are denoted by the vector  $\mathbf{x}_{\perp}$  and are called extended. The boundaries that constrain the medium on the sides are called the lateral boundaries. Pattern formation is defined to occur when there is spatial structure in the *extended directions*  $\mathbf{x}_{\perp}$ . It is most clearly understood when the description of the system is translationally invariant in the extended directions. Translational invariance can be achieved by assuming that the system is infinitely wide ( $L \rightarrow \infty$ ) or is spatially periodic in the extended coordinates  $\mathbf{x}_{\perp}$ .

## Examples:

- Rayleigh-Benard convection: heat flux from warm to cold boundary
- Taylor-Couette experiment: momentum flux from one moving cylinder to the other
- Reaction-diffusion systems: flow of chemicals from one porous wall to the other

Periodic boundary conditions amount to introduction of circular (in 1D) or toroidal (in 2D and 3D) topology. For instance, experiments in annular geometry allow one to study the dynamics without lateral boundaries. The figure on the left shows traveling pulses in binary convection.



In the following we will follow a two-step strategy:

- 1) consider the problem in the idealized geometries with translational invariance
- 2) investigate the small corrections of actual lateral boundaries

Note: Sometimes the effect of lateral boundaries is not small (e.g., traveling pulses might not be possible in a linear system, but are possible in a circular system), so one has to be careful.

We emphasize that the boundaries A and B are crucial in establishing the non-equilibrium state and cannot be idealized away via some assumption like translational invariance. As a result, *the structure of the state in the confined direction must be understood prior to the onset of pattern formation* via a proper formulation of the effect of these boundaries.

## Linear Stability Analysis of a Simple Pattern-Forming System

The Swift-Hohenberg equation

$$\partial_t u = ru - (1 + \partial_x^2)^2 u - u^3$$

was invented in the context of trying to understand the onset of Rayleigh-Bénard convection in a large-**aspect-ratio** (ratio of lateral size  $L$  to height  $d$ ) cell.

The parameter  $r$  plays a role similar to the Rayleigh number  $R$  in that increasing  $r$  eventually leads to instability and to pattern formation.

You can think of the field  $u(t,x)$  as the  $z$ -component of the fluid's velocity field  $v_z(t,x,y_0,z_0)$  at mid-height ( $z_0 = d/2$ ) and midway across ( $y_0 = L_y/2$ ) a long narrow convection experiment of dimensions  $L_x \times L_y \times d$  with  $L_x \gg L_y > d$ .

The coordinate  $x$  is an extended horizontal coordinate  $\mathbf{x}_\perp$  of the sort that we discussed in the previous section. There is no confined coordinate  $\mathbf{x}_\parallel$  since, in the derivation of this equation, the confined direction of the convection cell was eliminated (we will discuss this later).

As one can quickly verify, the *evolution equation is equivariant* with respect to

- 1) translations  $x \rightarrow x + a$  (the differential operator has constant coefficients)
- 2) reflection  $x \rightarrow -x$  (there are no odd spatial derivatives)

The *spatially uniform steady state*  $u_b = 0$  is also symmetric with respect to both operations.

Writing  $u = u_b + \delta u$  and linearizing about the base state  $u_b = 0$  we obtain

$$\partial_t \delta u = (r - 1 - 2\partial_x^2 - \partial_x^4) \delta u \equiv J[\delta u],$$

As a consequence of symmetry of the uniform state and the evolution equation, the Jacobian should commute with the translation operator (please check!) and so its eigenfunctions should be given by Fourier modes

$$\delta u_q(x, t) = a_q(t) e^{iqx},$$

where  $q$  is the (as yet undetermined) wavenumber.

It is easy to check that this Fourier mode  $\delta u_q$  is indeed an eigenfunction of  $J$  with the eigenvalue  $\sigma_q = r - (1 - q^2)^2$ :

$$J[\delta u_q] = \sigma_q \delta u_q,$$

and, therefore, the linearized evolution equation can be solved separately for each  $q$ :

$$a_q(t) = a_q(0) e^{\sigma_q t}.$$

In order to determine  $q$  we need to specify the lateral boundary conditions. There are two important cases:

- a) Laterally unbounded (infinite) system:  $\delta u_q$  has to be everywhere bounded, so any real  $q$  is possible.
- b) Periodic boundary conditions: for a system of lateral size  $L$ ,  $\delta u_q$  should be periodic

$$\delta u_q(x + L, t) = \delta u_q(x, t) \quad \Rightarrow \quad q = \frac{2\pi}{L} n, \quad n = 0, \pm 1, \pm 2, \dots$$

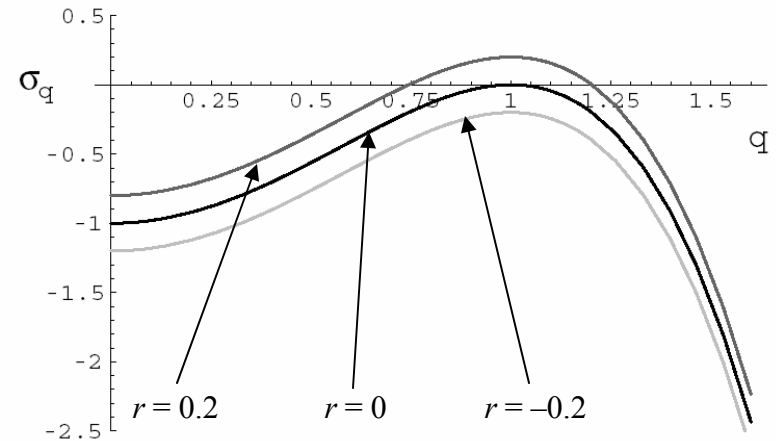
If we define  $\sigma_q = \gamma_q + i\omega_q$ , we can see that the Fourier mode

$$\delta u_q(x, t) = a_q(0) e^{\gamma_q t} (\cos \omega_q t + i \sin \omega_q t) e^{iqx}$$

oscillates with frequency  $\omega_q = \text{Im}(\sigma_q)$ , while its amplitude decays exponentially, if  $\gamma_q = \text{Re}(\sigma_q) < 0$  and grows exponentially otherwise. In order for the base state  $u_b = 0$  to be stable *all* eigenvalues should have negative real parts. (For the Swift-Hohenberg equation both  $q$  and  $\sigma_q$  are real.)

Let us suppose that the system is infinite, and plot  $\sigma_q$  as a function of wavenumber (we only need positive  $q$  as  $\sigma_q$  is even). The maximum growth rate  $\sigma_q = r$  occurs at  $q = 1$ , so for

- a)  $r < 0$ : all eigenvalues are negative and the uniform state is **stable**.
- b)  $r > 0$ : there are positive eigenvalues and the uniform state is **unstable**.



We can, therefore, conclude that at the **critical parameter value**  $r_c = 0$  a bifurcation occurs at which the uniform steady state  $u_b = 0$  loses stability.

The wavenumber of the mode that becomes unstable at critical parameter value  $r = r_c$  is called the **critical wavenumber**,  $q_c$ . (For the Swift-Hohenberg equation  $q_c = 1$ .)

Finally,  $\omega_{q_c}$  is called the **critical frequency**. (For the Swift-Hohenberg equation  $\omega_{q_c} = 0$ .)

What happens above threshold, when the uniform state has become unstable?

As the evolution equation for  $\delta u$  is *linear*, the general solution is given by a linear superposition

$$\delta u(x, t) = \int_q a_q(0) e^{\sigma_q t} e^{iqx} dq \quad (\text{infinite system})$$

$$\delta u(x, t) = \sum_q a_q(0) e^{\sigma_q t} e^{iqx} \quad (\text{periodic boundary conditions})$$

All modes with  $\gamma_q = \text{Re}(\sigma_q) < 0$  will eventually decay, so the growing pattern will be composed only of modes with  $\gamma_q > 0$ . For small  $r$  (i.e., close to onset of instability) these modes will all have  $q \approx q_c$  and hence  $\omega_q \approx \omega_{q_c}$ .

For many interesting classes of bifurcations (e.g., supercritical pitchfork bifurcation) nonlinear terms will be small, so their effect will be limited to **saturation of the growth** of these modes. One can, therefore, conclude that the critical wave number  $q_c$  and frequency  $\omega_{q_c}$  determine the characteristic **length scale**  $\lambda$  and the oscillatory **time scale**  $\tau$  of the arising pattern:

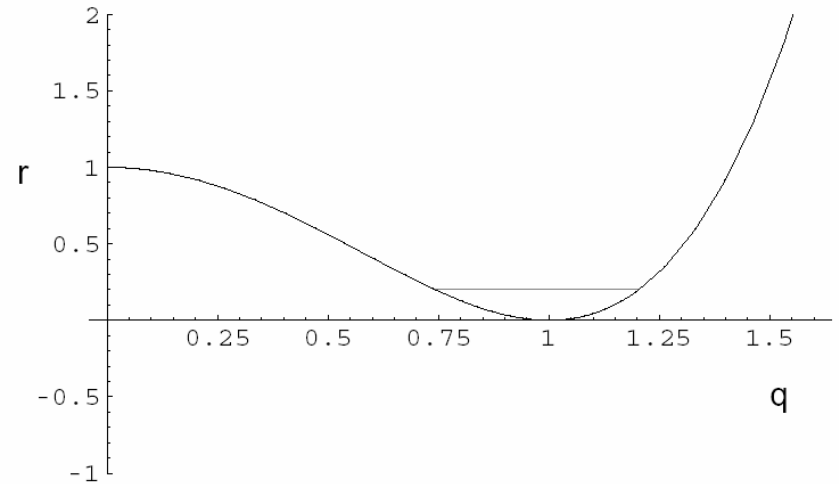
$$\lambda = \frac{2\pi}{q_c}, \quad \tau = \frac{2\pi}{\omega_{q_c}}.$$

For instance, for the Swift-Hohenberg equation,  $\lambda = 2\pi$  defines the wave length of convection rolls at the onset of convection, while  $\tau = \infty$  suggests that these rolls are stationary (it takes an infinitely long time for them to change).

An alternative and commonly used way to summarize the information about the spectral composition of the growing pattern is to plot the **neutral stability curve**  $r = r(q)$  for which the growth rate vanishes,  $\text{Re}(\sigma_q) = 0$ .

For the Swift-Hohenberg equation, the neutral stability curve for the uniform state  $u_b = 0$  (left) is given by the curve

$$r(q) = (1 - q^2)^2$$



For all points  $r < r(q)$  below this curve, the uniform state  $u = 0$  is linearly stable, so we would expect pattern formation to occur above this curve. You can think of this curve as telling us when a small-amplitude perturbation of wave number  $q$  will be stable or unstable as the parameter  $r$  is varied. Unlike the graph of  $\sigma_q$ , however, the neutral stability curve gives no information about the *growth rate* of the instability. The critical parameters  $r_c$  and  $q_c$  are now determined graphically by the *global* minimum of the neutral stability curve. The figure once again suggests that cellular patterns with wave number  $q \approx q_c$  might be expected for  $r$  just larger than  $r_c$  since there is a narrow band of wave numbers for which the uniform state is unstable.

The first bifurcation from the uniform state at  $r = r_c$  is usually called the **primary bifurcation** of the pattern forming system. Successive bifurcations leading to new states are sometimes called **secondary** and **tertiary** bifurcations.

## The key steps of a linear stability analysis:

- (i) Obtain explicitly the evolution equations for the system.
- (ii) Rewrite the evolution equations in dimensionless form to reduce the number of parameters  $\mathbf{p}$  and to obtain the parameters in dimensionless form.
- (iii) Replace the boundary conditions in the extended directions with infinite or periodic boundary conditions.
- (iv) For a given vector of system parameters  $\mathbf{p}$ , find explicitly at least one time-independent uniform state  $\mathbf{u} = \mathbf{u}_{unif}(\mathbf{x}_{\parallel})$  that is uniform with respect to the extended coordinates  $\mathbf{x}_{\perp}$ .
- (v) Linearize the evolution equations about the uniform state  $\mathbf{u}_{unif}$  to obtain the linear evolution equations for an infinitesimal perturbation  $\delta\mathbf{u}$ . The coefficients of these linear evolution equations will not depend on the extended coordinates  $\mathbf{x}_{\perp}$ .
- (vi) Use a particular solution of the form  $\delta\mathbf{u} = \mathbf{u}_{\mathbf{q}}(\mathbf{x}_{\parallel})e^{i\mathbf{q}\cdot\mathbf{x}_{\perp}}e^{\sigma_{\mathbf{q}}t}$  to solve the linearized evolution equations and to obtain the growth rate  $\sigma_{\mathbf{q}}$ .
- (vii) Analyze the function  $\text{Re}(\sigma_{\mathbf{q}})$  versus wave vector  $\mathbf{q}$  visually and mathematically. The interesting features to look for are local maxima (especially the global maximum) and the wave vectors  $\mathbf{q}$  corresponding to these maxima. The uniform state is linearly stable for the chosen parameters if  $\max_{\mathbf{q}} \text{Re}(\sigma_{\mathbf{q}}) < 0$ , unstable otherwise.
- (viii) Determine the region of stability (in the parameter space) of the uniform state  $\mathbf{u}_{unif}$  by solving the equation  $\max_{\mathbf{q}} \text{Re}(\sigma_{\mathbf{q}}) = 0$ . Identify the critical wave number  $q_c$  and critical frequency  $\omega_c = \text{Im}(\sigma_{\mathbf{q}})$  on the stability boundary.