(a) To nondimensionalize the equation, rescale the variables: $A = aA'$, $x = Xx'$, $y = Yy'$, and $t = Tt'$. This yields

$$\frac{\tau a}{T}\partial_{t'}A' = \epsilon a A' + \frac{\xi_x^2 a}{X^2}\partial_{x'}^2 A' + \frac{\xi_y^2 a}{Y^2}\partial_{y'}^2 A' - g a^3 |A'|^2 A'$$

or after multiplying by $T/a\tau$ and dropping the primes

$$\partial_t A = \frac{\epsilon T}{\tau} A + \frac{\xi_x^2 T}{X^2 \tau}\partial_x^2 A + \frac{\xi_y^2 T}{Y^2 \tau}\partial_y^2 A - \frac{g a^2 T}{\tau}|A|^2 A.$$

Setting all the prefactors to unity and solving the resulting equations yields

$$T = \frac{\tau}{\epsilon}, \qquad X = \frac{\xi_x}{\epsilon^{1/2}}, \qquad Y = \frac{\xi_y}{\epsilon^{1/2}}, \qquad a = \frac{\epsilon^{1/2}}{g^{1/2}}.$$

In terms of the new variables

$$\partial_t A = A + \nabla^2 A - |A|^2 A.$$

(b) Linearizing the rescaled equation about a uniform state, we find

$$\partial_t \delta A = \delta A + \nabla^2 \delta A - 2|A|^2 \delta A - A^2 \delta A^*.$$

For the trivial state $A = 0$

$$\partial_t \delta A = \delta A + \nabla^2 \delta A,$$

so in principle we could extrapolate from the real variable case to find the growth rate $\sigma = 1 - Q^2$ for perturbations $\delta A \propto e^{i\mathbf{Q}\cdot\mathbf{x}} e^{\sigma t}$. In terms of the real and imaginary components $\delta A = U + iV$ we have

$$\partial_t (U + iV) = U + iV + \nabla^2 (U + iV).$$

Collecting the real and imaginary parts, we find

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 + \nabla^2 & 0 \\ 0 & 1 + \nabla^2 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}$$

or, upon substitution of $U = U_Q e^{i\mathbf{Q}\cdot\mathbf{x}} e^{\sigma t}$ and $V = V_Q e^{i\mathbf{Q}\cdot\mathbf{x}} e^{\sigma t}$,

$$\sigma \begin{pmatrix} U_Q \\ V_Q \end{pmatrix} = \begin{pmatrix} 1 - Q^2 & 0 \\ 0 & 1 - Q^2 \end{pmatrix} \begin{pmatrix} U_Q \\ V_Q \end{pmatrix}.$$

Since the matrix is diagonal, we again find $\sigma = 1 - Q^2$. Clearly, the system is unstable for all $\epsilon > 0$ for which the rescaled amplitude equation we derived is valid.

(c) $A(x, y, t)$ is just an amplitude describing a pattern in the extended directions (x, y). The pattern can only arise if there is a gradient of some quantity in a confined direction driving the system out of equilibrium. Hence, there should be at least one confined direction, e.g., z, such that $\mathbf{u} = \mathbf{u}(x, y, z, t)$.

(d) The amplitude equation applies only to the type-I_s instability in a uniaxial (nonisotropic) system, so the u-field should undergo a type-I_s instability and the system should be uniaxial. The instability type for the amplitude equation is type-III_s, since the maximum of σ is achieved at $Q = 0$.

Even though the instability types for $\mathbf{u}$ and A are different, there is no inconsistency, since $\mathbf{u} = Ae^{i\mathbf{q}_c\cdot\mathbf{x}} + c.c. + h.o.t$ and therefore $\mathbf{q} = \mathbf{q}_c + \mathbf{Q} \neq 0$ when $\mathbf{Q} = \mathbf{Q}_c = 0$.

(e) This is a second order PDE, so it needs a pair of boundary conditions in each direction (one per boundary). The leading order in ϵ boundary condition is $A = 0$, so we should have $A = 0$ at $x = \pm L_x$ and $y = \pm L_y$.

(f) In the scaled variables the distance is $O(1)$, since there are no parameters in the scaled equation. Therefore, in the original variables the distance is $x_0 = \xi_x/\epsilon^{1/2}$ and $y_0 = \xi_y/\epsilon^{1/2}$, both of which are $O(\epsilon^{-1/2})$.

(g) In a uniaxial system the stripes could form only along x or y direction. For from the boundaries we can write $A_K = A_s e^{iKx}$ (or $A_K = A_s e^{iKy}$) such that in steady state $0 = A_s - K^2 A_s - |A|^2 A_s$ and threfore $A_s = \sqrt{1-K^2}e^{i\phi}$, where the phase ϕ is an arbitrary constant that defines the relative position of the stripe pattern.

In the original variables $A_s = (\epsilon/g)^{1/2}\sqrt{1-K^2}e^{i\phi}$, where $K = (q-q_c)\xi_x/\epsilon^{1/2}$ for a pattern with stripes along the y-direction or $K = (q-q_c)\xi_y/\epsilon^{1/2}$ for a pattern with stripes along the x-direction.

(h) We need to determine the stability of a stripe state $A_K(x)$ with respect to long-wavelength disturbances $\delta A(x, y, t)$. Following problem 1 of the homework assignment 9, we write

$$A(x, y, t) = A_K(x) + \delta A(x, y, t),$$

where

$$\delta A \sim e^{iKX}\left[a_+(t)e^{i\mathbf{Q}\cdot\mathbf{X}} + a_-^*(t)e^{-i\mathbf{Q}\cdot\mathbf{X}}\right]. \tag{1}$$

Linearizing the amplitude equation about $A_K(x)$, we find the following system of equations for $a_\pm$:

$$\partial_t \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = -\begin{pmatrix} P^2 + U_+ & P^2 \\ P^2 & P^2 + U_- \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix},$$

where $P^2 = 1 - K^2$ and $U_\pm = Q_x^2 + Q_y^2 \pm 2KQ_x$.

For $Q_x = 0$ the leading eigenvalue is $\sigma = -Q_y^2 < 0$, so the stripes are stable relative to the zigzag instability for all K. For $Q_y = 0$ the leading eigenvalue is

$$\sigma = -\frac{1-3K^2}{1-K^2}Q_x^2 + O(Q_x^4)$$

so that the stripes are stable with respect to the Eckhaus instability for $|K| < 1/3$ and unstable otherwise.

In original variables, the pattern $\mathbf{u}(x) \propto e^{iqx}$ would be stable when $|q - q_c| < \epsilon^{1/2}\xi_x^{-1}/3$ and the pattern $\mathbf{u}(y) \propto e^{iqy}$ would be stable when $|q - q_c| < \epsilon^{1/2}\xi_y^{-1}/3$.

Strictly speaking, this result holds only for ϵ small, i.e., close to onset. Also, this result determines only when the stripes are *unstable*. In the region where the amplitude equation predicts stability, the stripes could still be unstable with respect to a different (short-wavelength) instability.