Problem 1

Applications of the Reynolds Number. For problems in which an isothermal fluid flows through a pipe or past an object like a cylinder, an analysis of the Navier-Stokes equations reveals a dimensionless stress parameter called the Reynolds number R,

$$R = \frac{vL}{\nu},$$

where v is a characteristic magnitude of the fluid's velocity field (say the maximum speed of the fluid before it encounters some obstacle), L is the size of the object with which the fluid interacts (e.g., the diameter of the pipe or of the cylinder), and ν is the kinematic viscosity of the fluid, the same parameter that appears in the Rayleigh number (see lecture notes). For small flow speeds corresponding to $R < 1$, the fluid is usually laminar, i.e., time independent and without an interesting spatial structure (the stream lines are approximately parallel). For Reynolds numbers larger than about 1, laminar flows usually become unstable and some new kind of patterns or dynamics occur. When R becomes larger than about 1000, the fluid flow often becomes chaotic in time and irregular in space. The following questions give you a chance to appreciate the many useful predictions that can be made by studying a parameter like the Reynolds number.

(a) Show that the Reynolds number is a dimensionless quantity and so has the same value in any system of units.

(b) For an airplane traveling at $v = 500$ km/hour, will the airflow over the wing be laminar or turbulent?

(c) As you walk around a room, show that the air in the vicinity of your foot will be turbulent. This implies that a cockroach will need some way to locate your foot in the midst of a turbulent flow to avoid being stepped on.

(d) By flipping a coin, estimate the speed with which it falls and the speed with which it rotates and then determine whether fluid turbulence plays a role in the supposedly random behavior of flipping a coin to call heads or tails.

(e) By using the size of the heart, the cross-section of an artery and the rate of heartbeat estimate the typical speed of blood flowing through your arteries and through your heart. Does the blood flow in any of your arteries become turbulent? What about through a heart valve (a couple centimeters across)? (The blood viscosity is about 3-4 times that of water.)

(f) When the wind blows transversely past a telephone wire, you sometimes hear an eerie whistling sound called an aeolian tune. Using the kinematic viscosity of air at room temperature from the table in the lecture notes and a wire diameter of $L = 2$ mm, what is the smallest wind velocity for which you would expect to hear an aeolian tune?

Problem 2

Linear stability and bifurcations. Find the steady states and their stability for the following simple dynamical systems. Draw the steady states as functions of the parameter r on the same diagram using a solid line for stable states and a dashed line for unstable ones.

(a) saddle-node bifurcation: $\dot{x} = r - x^2$

(b) supercritical pitchfork bifurcation: $\dot{x} = rx - x^3$

(c) subcritical pitchfork bifurcation: $\dot{x} = rx + x^3$

(d) transcritical bifurcation: $\dot{x} = rx - x^2$

Problem 3

Linearization of operators. Linearize (find the Jacobian operator for) the following nonlinear operators

(a) $F[u] = u\partial_x u$, with $u(x)$ – a scalar field, at $u_0(x) = u_0$.

(b) $F[u] = \nabla^2 u^2$, with $u(x, y)$ – a scalar field, at $u_0(x, y) = u_0$.

(c) $F[\mathbf{v}] = (\mathbf{v} \cdot \nabla)\mathbf{v}$, with $\mathbf{v}(x, y)$ – a two-dimensional vector field, at $\mathbf{v}_0(x, y) = \mathbf{v}_0$.

Recall that the Jacobian is the lowest-order term in the difference $F[v + \delta v] - F[v]$ in the limit of small δv.