Problem 1

Exact solution for a moving front. Compute analytically the shape and speed of the moving front connecting the saturated stripe solution (with critical wavenumber) with the uniform state for the (complex) type-I$_s$ amplitude equation. *Hint: Use the fact that* $f(x) = \tanh(x)$ *is a solution of the differential equation:*

$$f'' + 2f - 2f^3 = 0 \tag{1}$$

Give the answer in dimensional (unscaled) form.

Problem 2

Pulled Fronts in the Swift-Hohenberg Equation.

(a) Show that for the Swift-Hohenberg equation

$$\partial_t u(x,t) = ru - (\partial_x^2 + 1)^2 u - u^3 \tag{2}$$

the velocity c of a pulled front and the stationary-phase wave number q_s are given by

$$c = \frac{4}{3\sqrt{3}}\left(2 + \sqrt{1+6r}\right)\left(-1 + \sqrt{1+6r}\right)^{1/2} \tag{3}$$

and

$$q_s = \frac{1}{2}\left(3 + \sqrt{1+6r}\right)^{1/2} + \frac{i}{2\sqrt{3}}\left(-1 + \sqrt{1+6r}\right)^{1/2} \tag{4}$$

(b) For small r check whether these results agree with the results expected from the amplitude equation approach.

(c) Show that the wave number of the pattern laid down behind the advancing front, assuming no phase slips, is given by

$$q_{fp} = \frac{3\left(3+\sqrt{1+6r}\right)^{3/2}}{8\left(2+\sqrt{1+6r}\right)} = 1 + \frac{r}{8} + \cdots. \tag{5}$$

Problem 3

Qualitative analysis of fronts in a nonlinear diffusion equation. In this exercise you will investigate front propagation in the real nonlinear diffusion equation

$$\partial_t u(x,t) = \partial_x^2 u + F(u), \qquad F(u) = \epsilon u + u^3 - u^5. \tag{6}$$

Note that this equation corresponds to an amplitude equation for a subcritical bifurcation restricted to real solutions.

(a) Consider front solutions $u(\xi)$ with $\xi = x - ct$. Show that the equation for the front can be written as

$$\begin{aligned} u' &= v, \\ v' &= -F(u) - cv \end{aligned} \tag{7}$$

where primes denote $d/d\xi$. We will study (7) as a dynamical system with ξ acting as the fictitious time.

(b) Show that for $\epsilon > -1/4$ there is a fixed point $(u, v) = (u_0, 0)$ with $u_0 \neq 0$ that corresponds to the saturated stationary nonlinear state of equation (6) that is temporally stable.

(c) Analyze the stability of this fixed point within the dynamical system (7), and show that it has one stable direction and one unstable direction for any positive value of c.

(d) Establish the stability of the fixed point $(u, v) = (0, 0)$ of the dynamical system (7), and the number of its stable and unstable directions, for all c and ϵ.

(e) For $\epsilon > 0$ and $c > 2\sqrt{\epsilon}$ sketch a trajectory in the (u, v) plane corresponding to a "pulled" front of the nonlinear solution advancing into the $u = 0$ solution towards positive x, being careful to show how the trajectory behaves near each fixed point (i.e., its direction relative to the various eigenvectors). Use this to argue that if such a front exists then it is a member of a continuous family.

(f) Sketch the trajectory corresponding to a "pushed" front.

(g) By considering the trajectories linking the two fixed points for $-1/4 < \epsilon < 0$ argue that the value of c must be tuned to find a front solution, i.e., there is a unique front propagation speed in this case. Note that this is the case of a front propagating into a temporally stable state.

(h) Can you show that the speed of the "pushed" front for $\epsilon > 0$ is continuous, while for $\epsilon < 0$ there is a unique front speed?