Problem 1

Piecewise Linear Model. Consider the reaction-diffusion system

$$\begin{aligned}\partial_t u &= \eta^{-1} f(u,v) + \nabla^2 u,\\ \partial_t v &= g(u,v),\end{aligned} \tag{1}$$

with a "piecewise linear" form of the reaction kinetics:

$$\begin{aligned}f(u,v) &= \theta(u-a) - u - v,\\ g(u,v) &= u - bv,\end{aligned} \tag{2}$$

where $\theta(\cdot)$ is a Heaviside step function and $\eta \ll 1$.

(a) Sketch the u and v nullclines for $a = 0.25$ and (i) $b = 0.2$ and (ii) $b = 1$. For what ranges of b is the reaction kinetics monostable and bistable for this value of a.

(b) Consider the front connecting, for $a = 0.3$, the rest state $u = 0$, $v = 0$ with the excited state at this value of v, i.e. $u = 1$, $v = 0$. Plot the effective potential $\Phi(u)$.

(c) What is the value of v for which the front connecting the small u and large u portions of the u-nullcline is stationary for $a = 0.25$ (i.e. $v = v^*$ giving the stall solution)?

(d) Calculate as a function of the parameter a the speed of the front connecting the rest state $u = 0$, $v = 0$ with the excited state at this value of v, i.e. $u = 1$, $v = 0$. Verify that in the approximation we used this front speed is zero for $a = 0.5$.

(e) Find the propagation speed c and calculate and plot $u(x-ct)$ and $v(x-ct)$ for the excitation pulse propagating in the rest state of this system with $a = 0.25$, $b = 0.2$.

(f) Calculate and plot expressions for the dispersion relationship $C(T)$ for the scaled speed $C = \eta^{1/2}c$ as a function of the temporal period T for propagating wave trains for $a = 0.25$, $b = 0.2$. Do not worry about the breakdown of the scaling that occurs for small C.