Problem 1

Type of Linear Instability for a Swift-Hohenberg Equation with a Second-Order Time Derivative. Consider a two-dimensional Swift-Hohenberg equation with a second-order time derivative:

$$\partial_t^2 u(x,y,t) = ru - (1+\nabla^2)^2 u - u^3.$$

Show that the base solution $u_0 = 0$ becomes linearly unstable at $r = 0$ and determine the type of instability (I_s, I_o, III_o, etc).

Problem 2

Type of Instability for Two Coupled Reaction-Diffusion Equations. Consider the following reaction-diffusion evolution equations

$$\partial_t u = au + \partial_x^2 u - b\partial_x^2 v - (u^2+v^2)(u+cv),$$
$$\partial_t v = av + b\partial_x^2 u + \partial_x^2 v - (u^2+v^2)(v-cu)$$

for the scalar fields $u(x,t)$ and $v(x,t)$ in one dimension, where the parameters a, b, and c are arbitrary real constants. (You can think about the fields u and v as representing concentrations of two reacting and diffusing chemicals.) Derive a condition for the linear instability of the zero base solution $(u_0, v_0) = (0,0)$ and determine whether the instability is of type I_s, I_o, or III_o. Hint: When you substitute the infinitesimal perturbation with the correct spatial and temporal dependence (the same for both fields) into the linearized PDEs, you will obtain an eigenvalue problem involving a 2×2 matrix. The eigenvalues of the matrix will give you the growth rates, while the eigenvectors will determine the corresponding relative amplitudes of the two fields.

Problem 3

Linear Stability Analysis in a Finite Geometry. Let us consider the one-dimensional Swift-Hohenberg equation for a field $u(x,t)$,

$$\partial_t u = ru - (1+\partial_x^2)^2 u - u^3,$$

on the finite interval $[-L/2, L/2]$ with the boundary conditions

$$u = \partial_x u = 0, \qquad \text{at } x = -L/2 \text{ and } x = L/2. \tag{1}$$

(The differential operator on the right-hand-side of the Swift-Hohenberg equation is fourth-order, so we need four boundary conditions to uniquely determine the solution.) The boundary condition $u = 0$ is similar to the no-slip velocity boundary condition $\mathbf{v} = 0$ for Rayleigh-Bénard convection in that it constrains the solution $u(x,t)$ at the boundaries toward the conducting state $u = 0$.

(a) Show that an infinitesimal perturbation $\delta u(x,t)$ of the zero homogeneous state $u = 0$ satisfies the following linear evolution equation:

$$\partial_t \delta u = \left(r - (1+\partial_x^2)^2\right)\delta u. \tag{2}$$

(b) We need to solve Eq. (2) on the interval $[-L/2, L/2]$, which means that we need to specify boundary conditions on δu to identify a unique solution. Explain why the perturbation $\delta u(x,t)$ must also satisfy the boundary conditions (1) which, upon substitution $u \to \delta u$, become the boundary conditions for Eq. (2).

(c) We can use separation of variables in Eq. (2) to determine that the time dependence of any solution δu is exponential in time with growth rate σ:

$$\delta u = e^{\sigma t} f(x),$$

where $f(x)$ is the eigenfunction corresponding to σ. The state $u = 0$ is marginally unstable when $\sigma = 0$ which means that δu is time-independent, which further means that the eigenfunction f of the marginally stable mode must satisfy the time-independent equation:

$$(1 + \partial_x^2)^2 f = r f, \tag{3}$$

with boundary conditions $f = \partial_x f = 0$ given by (1). Note that Eq. (3) is itself an eigenvalue problem: for a given system size L, solutions f can be found only for special values of the parameter r. Alternatively, for a given value of r, nonzero solutions can only be found for a special choice of length L.

Since Eq. (3) is an equation with constant coefficients, the solutions are sinusoidal, and with the symmetric formulation of the problem about $x = 0$, the solutions will either be even (pure cosine) or odd (pure sine). Substituting $f = \cos(qx)$ or $f = \sin(qx)$ into Eq. (3), show that q must take on one of two values:

$$q_{\pm} = \sqrt{1 \pm \sqrt{r}}.$$

(d) The previous result and linearity suggests that the general symmetric solution of Eq. (3) will be of the form:

$$f(x) = \begin{cases} A_+ \cos(q_+ x) + A_- \cos(q_- x), & \textit{even} \\ A_+ \sin(q_+ x) + A_- \sin(q_- x), & \textit{odd} \end{cases} \tag{4}$$

where A_+ and A_- are constants. Use Eq. (4) and the boundary conditions on f to derive a transcendental equation $g(r, L) = 0$ that relates the parameters r and L.

(e) For a given system size L, the onset of convection will correspond to the first positive root of the transcendental equation that you derived. Show that for large system sizes L, the first positive root is given approximately by:

$$r_c \approx \left(\frac{2\pi}{L}\right)^2. \tag{5}$$

Graphical plots of your transcendental equation may help you to see what kind of approximations are needed to get this result.

Eq. (5) answers the original question of how finite boundaries modify the onset of a type-I_s instability for boundary conditions corresponding to those of a fluid. It is harder to initiate the instability in a finite system and the critical parameter value increases roughly as $1/L^2$ as the system size L is decreased. This is a specific prediction that can be tested experimentally.

(f) For a length $L = 10$, calculate numerically the critical value r_c to three digits, compare this answer with Eq. (5), and plot the corresponding eigenfunction (4). You are welcome to use Mathematica, Maple, or any other similar program to facilitate these (and any other) calculations.