Problem 1

Identification of a Dimensionless Stress Parameter. To see how a dimensionless stress parameter like the Rayleigh number might be discovered from known dynamical equations, consider the evolution equation for a damped driven pendulum,

$$m\ddot{\theta} + \alpha\dot{\theta} + C\sin(\theta) = A\sin(\omega_0 t), \tag{1}$$

where m is the mass of the pendulum, $\theta(t)$ is the angle of the pendulum with respect to the vertical (so $\theta = 0$ means the particle hangs down vertically underneath the fulcrum), α is the damping coefficient, A is the amplitude of the external driving, ω_0 is the frequency of the external driving, and overhead dots denotes differentiation with respect to time, e.g., $\ddot{\theta} = d^2\theta/dt^2$. One parameter in this equation can be eliminated immediately by dividing both sides of Eq. (1) by m, which redefines $m = 1$ and the other parameters to the values α/m, C/m, and A/m. A second parameter can be eliminated by defining a new dimensionless time coordinate $\bar{t}$ by the transformation:

$$t = c_t\bar{t},$$

where c_t is a new unit of time.

(a) Write down Eq. (1) in the new variable $\bar{t}$ by substituting and using the chain rule of calculus.

(b) Describe the three possible ways to choose the value of c_t so as to eliminate a second coefficient in the new equation by setting the coefficient equal to one. Discuss also the physical meaning of the three different choices of c_t. For example, one choice is $c_t = m/\alpha$ which is the time scale for exponential decay arising from the dissipative term $\alpha\dot{\theta}$.

(c) Only one of your three choices of time scale c_t leads to a Rayleigh-like parameter with the meaning of "ratio of strength of driving to strength of dissipation." What is that choice and what is the resulting dimensionless stress parameter?

(d) The dimensionless quality factor Q of an oscillator measures the sharpness (width) of the resonant response of an oscillator to small amplitude driving as the driving frequency ω_0 is varied. Find a dimensionless combination of the physical parameters that could correspond to Q.

(e) Discuss whether it would be useful to use a rescaled angle variable $\bar{\theta} = \theta/c_\theta$ to further simplify the equation.

Problem 2

Scaling of Time, Length, and Magnitude Scales for the Swift-Hohenberg Equation. To gain further experience simplifying a dynamical equation and identifying dimensionless parameters, repeat the previous exercise but now for the Swift-Hohenberg equation for a real-valued field $u(t,x)$ that depends on time t and one spatial coordinate x:

$$\tau_0\partial_t u = ru - \xi_0^4(q_0^4 + 2q_0^2\partial_x^2 + \partial_x^4)u - g_0u^3. \tag{2}$$

Eq. (2) seems to have five distinct parameters, namely the time scale τ_0, the coherence length ξ_0, the critical wave number q_0, the nonlinear strength g_0, and the control parameter r.

By a clever choice of time, length and magnitude scales t_0, x_0 and u_0, i.e., by changing variables from t, x, and u to the scaled variables τ, y, and v by the equations

$$t = t_0\tau, \qquad x = x_0 y, \qquad u = u_0 v,$$

and by redefining the parameter r to a new value $\hat{r}$, show that Eq. (2) can be written in a dimensionless form with only one parameter:

$$\partial_\tau v = \hat{r}v - (1 + 2\partial_y^2 + \partial_y^4)v - v^3.$$

This is a substantial simplification since the mathematical and numerical properties of this equation can be explored as a function of a single parameter $\hat{r}$.

Problem 3

Non-Dimensionalization of the Boussinesque Equations. Consider the Boussinesque equations for the disturbances P, θ and $\mathbf{v}$ about the conduction profile. We would like to reduce the number of parameters in the equations by rescaling these disturbances along with the indepdendent variables $\mathbf{x}$ and t. (Note: the three spatial coordinates enter the equations in a rather symmetric way, so we will not gain much by introducing different length scales in the x, y and z directions.)

(a) Before doing any calculations, determine how many parameters and how many prefactors there are in the three Boussinesque equations, and how many scales (to be determined) we can introduce to facilitate the non-dimensionalization procedure. Given those numbers predict the minimal number of dimensionless parameters that have to be specified to completely describe the system.

(b) Now do the calculations. Introduce the scales for all dependent and independent variables and rewrite the equations in terms of the new (dimensionless) variables. Make sure all terms in these equations are dimensionless.

(c) Using these equations identify all dimensionless combinations of dimensional parameters that enter as prefactors in front of different terms. How many such prefactors (not equal to unity) did you get?

(d) We cannot set all these dimensionless prefactors to unity by choosing the scales (too many prefactors, too few scales), so select the prefactors that will be set to unity such that the non-dimensionalized Boussinesque equations in the lecture notes would be obtained. Use the resulting equations to determine all the scales. Check that the scales have correct units.

(e) Finally, use the scales that you have just found to determine the Rayleigh number and the Prandtl number by evaluating the remaining dimensionless prefactors in the Boussinesque equations.