Problem 1

Hexagonal patterns. Consider the superposition of modes

$$u(x, y) = e^{i(\mathbf{q}_1 \cdot \mathbf{x} + \phi_1)} + e^{i(\mathbf{q}_2 \cdot \mathbf{x} + \phi_2)} + e^{i(\mathbf{q}_3 \cdot \mathbf{x} + \phi_3)} + c.c.$$

with $\mathbf{x} = (x, y)$ and the wave vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ forming an equilateral triangle

$$\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 = 0, \qquad |\mathbf{q}_i| = 1$$

and where the ϕ_i are the possible phases of the modes. Since q sets the scale of the pattern, let's choose $q = 1$, and orient our axes so that $\mathbf{q}_1 = (1, 0)$.

(a) Show that by redefining the origin of coordinates we may set ϕ_1 and ϕ_2 to zero, so that the from of the pattern (outside of translations and rotations) is determined by a single phase variable $\phi_3' = \phi$.

(b) Using Mathematica, Matlab, or some other convenient plotting environment make some contour or density plots of the field $u(x, y)$ for various choices of ϕ. Do you get hexagonal patterns for an arbitrary choice of ϕ?

(c) As we will discuss later, the amplitude equation analysis near the onset of the instability shows that the sum of the phases $\phi_1 + \phi_2 + \phi_3$ (and so the reduced phase ϕ) will asymptotically evolve towards 0 or π. Plot the patterns for these two values. Describe the locations of the maxima and minima of the two patterns.

(d) How are the patterns for $\phi = 0$ and $\phi = \pi$ related in this approximation?

Problem 2

Hexagonal superlattice. As you have seen in class, a hexagonal superlattice can be constructed via a linear superposition of 12 complex exponentials with wave vectors lying on a triangular lattice aligned with a critical circle $q = q_c$.

(a) How many component stripes form the pattern and what are the angles between these stripes?

(b) Show that all 12 base wave vectors as well as their sums and differences can be represented by linear superpositions (with integer coefficients) of just two vectors, $\mathbf{q}_1$ and $\mathbf{q}_2$. Find the magnitude and mutual orientation of $\mathbf{q}_1$ and $\mathbf{q}_2$.

(c) Show that the pattern produced is indeed periodic and find its period.

Problem 3

Quasicrystalline patterns. Write down the superposition of modes with wave number $q = 1$ giving quasicrystalline states with 8-fold and with 10-fold rotational symmetry. Using Mathematica, Matlab, or some other convenient plotting environment plot the resulting patterns.