Problem 1

Nonlinear saturation in the Swift-Hohenberg equation.
Consider the Swift-Hohenberg equation

$$\partial_t u = ru - (1+\partial_x^2)^2 u - u^3 \tag{1}$$

(a) Repeat the calculation for the nonlinear saturated stripe state using Galerkin expansion truncated to one mode for a general wave vector, i.e., calculate a_q in the expression

$$u = a_q \cos qx + \cdots$$

(Note: it is helpful to understand how large $q - q_c$ is compared to r.)

(b) Repeat the calculation for the nonlinear saturated stripe state using Galerkin expansion truncated to three modes for the critical wave vector $q = q_c = 1$, i.e., calculate a_1, a_3 and a_5 in the expression

$$u = a_1 \cos x + a_3 \cos 3x + a_5 \cos 5x + \cdots$$

In principle we could immediately substitute this expansion into the Swift-Hohenberg equation with $\partial_t u = 0$. Setting the coefficients of three lowest order harmonics to zero would then provide the equations that can be solved for a_1 through a_5. However, analytical solution of this system is difficult due to the unknown relative magnitude of the coefficients. Instead let us follow a more systematic approach.

(i) Each coefficient a_n is a smooth function of r, so we can expand the coefficients in the following way:

$$a_n = A_{n1} r^{1/2} + A_{n2} r + A_{n3} r^{3/2} + A_{n4} r^2 + A_{n5} r^{5/2} + \cdots$$

Substitute the resulting expansion

$$u = \sum_{mn} A_{mn} r^{n/2} \cos mx$$

into the equation (1).

(ii) To simplify the task of finding the Fourier coefficients recall that $\cos mx$ and $\cos nx$ are orthogonal for $m \neq n$. The Fourier coefficients can therefore be found by multiplying the equation (1) by $\cos mx$ and integrating over $0 < x < 2\pi$.

(iii) Setting the Fourier coefficients of the lowest harmonics to zero we will obtain equations of the form

$$F_{m1}(A) r^{1/2} + F_{m2}(A) r + F_{m3}(A) r^{3/2} + \cdots = 0,$$

where $F_{mk}(A)$ are multinomials in $A_{11}, A_{12}, \cdots, A_{31}, \cdots$. Write down these equations for $m = 1, 3, 5$, keeping terms up to order $r^{7/2}$.

(iv) Each Fourier coefficient should vanish for all r, yielding a set of equations

$$F_{mk}(A) = 0, \qquad k = 1, 2, 3, \cdots$$

Solve these equations iteratively to find A_{mk} with $m = 1, 3, 5$ and $k = 1, \cdots, 5$, thereby obtaining the expression for the saturated state $u(x)$ up to terms of order $r^{5/2}$.

Note: You are encouraged to use Maple or Mathematica to do the calculations. Doing such calculations on paper are (1) a waste of your time and (2) a potential source of algebraic errors.

Problem 2

Swift-Hohenberg equation with broken up-down symmetry. In simple bifurcation theory we associate a pitchfork bifurcation with a system that is symmetric under the change in sign of the dynamical variable e.g., $u \to -u$. Consider however the generalized Swift-Hohenberg equation for $u(x,t)$ in one spatial dimension

$$\partial_t u = ru - (1+\partial_x^2)^2 u - gu^2 - u^3$$

which for nonzero g does not have this symmetry.

(a) By considering the ansatz

$$u = \sum_n a_n \cos nx$$

with the sum over all n (even and odd) show the bifurcation remains of pitchfork character with $a_1 = \pm\sqrt{4r/3}$ to lowest order.

(b) Calculate a_2 to lowest nontrivial order.

(c) Show that the two nonlinear solutions are not in general related simply by a change of sign.

Note that in two dimensions a transcritical transition to a hexagonal state is indeed obtained for nonzero g.

Problem 3

Lorenz equations. Repeat the analysis we have done for the porous medium (d'Arcy) convection now for the case of free liquid (Rayleigh-Bénard) convection with free slip boundary conditions, hence deriving the Lorenz equations.