– You are highly encouraged to use Maple or Mathematica to do these problems.
– You are welcome (and encouraged) to use and modify the Maple program(s) posted on the class homepage.

Problem 1

Quintic nonlinearity. Consider a variation of the Swift-Hohenberg equation with a quintic nonlinearity

$$\partial_t u = ru - (1 + \nabla^2)^2 u - u^5,$$

where $u(x, y, t)$ is a real field in a two-dimensional space.

(a) Conduct the linear stability analysis and determine the type of the primary instability.

(b) The next several questions consider base patterns with wave numbers equal to the critical one, $q = q_c$. Find the nonlinear saturated *stripe* solution. You will need to determine which harmonics will be present in the Galerkin expansion and find the coefficients of the three leading terms.

(c) Find the nonlinear saturated *square* solution. You may stop after you find the leading order term in the Galerkin expansion.

(d) Find the nonlinear saturated *hexagon* solution. You may stop after you find the leading order term in the Galerkin expansion.

(e) Determine the linear stability of the stripe state with respect to a cross-roll instability where the new set of stripes can have arbitrary orientation and wave number.

(f) Determine the linear stability of squares with respect to stripes.

(g) Determine the linear stability of hexagons with respect to stripes.

(h) Use the results of parts (e)-(g) to determine which pattern (stripes vs. squares vs. hexagons) will be selected at onset of primary instability?

(i) The remaining questions consider the *stripe* pattern with wave number *not* equal to critical, $q \neq q_c$. Find the leading order term in the Galerkin expansion for the nonlinear saturated solution.

(j) Test the stability of the saturated stripe state with respect to the cross-roll instability where the new set of stripes can have arbitrary orientation and wave number.

(k) Test the stability of the saturated stripe state with respect to the zigzag instability.

(l) Test the stability of the saturated stripe state with respect to the Eckhaus instability.

(m) Use the results of parts (j)-(l) to plot the stability balloon for saturated stripe state. On your diagram sketch the neutral stability curve and the curves which correspond to the above three secondary instabilities.

(... turn over for Problem 2)

Problem 2

Pattern selection in the absence of $u \to -u$ symmetry. Consider a variation of the Swift-Hohenberg equation with both a quadratic and a cubic nonlinearity

$$\partial_t u = ru - (1 + \nabla^2)^2 u - gu^2 - u^3,$$

where $u(x, y, t)$ is a real field in a two-dimensional space and $g \ll 1$ is a constant. Furthermore, we assume r to be non-negative but very small, that is $0 \leq r \ll g^2$, so that all expressions involving radicals can be expanded in power series where only the lowest powers of r should be kept.

(a) We have already performed the linear stability analysis of this equation and computed the saturated stripe solution (see assignment #6). If you haven't done it then, do it now. Only keep the leading order term(s). Here and below we will only consider patterns with the critical wave number, $q = q_c$.

(b) Find the nonlinear saturated *square* solution. You may stop after you find the leading order term(s) in the Galerkin expansion.

(c) Find the nonlinear saturated *hexagon* solutions (you should get two *different* solutions for the amplitude of the hexagons). You may stop after you find the leading order term in the Galerkin expansion. Do *not* expand the radicals in power series in this part.

What type of bifurcations (sub- or super-critical pitchfork, transcritical or other) is experienced by each of these two solutions and at what values of r? Sketch the amplitude of the component stripes for all solutions you found in parts (a)-(c) and indicate where and which bifucations you expect for different emerging patterns. We will check these predictions below.

(d) Determine the linear stability of the stripe state with respect to a cross-roll instability where the new set of stripes can have arbitrary orientation.

(e) Determine the linear stability of squares with respect to stripes.

(f) Determine the linear stability of hexagons with respect to stripes. You should do the linear stability analysis for each of the two hexagon solution separately.

(g) Can you use the results of parts (d)-(f) to determine which pattern (stripes vs. squares vs. hexagons) will be selected at onset of primary instability? Why?

(h) Now we do a more careful analysis. Find the three coupled nonlinear evolution equations for the amplitudes of the three sets of stripes composing a hexagon solution.

(i) Find the fixed points (or steady states) of this system of equations corresponding to the hexagon solution(s) and stripe solution, linearize the system about each of these steady states and determine their linear stability.

(j) Use these results to determine which pattern will be selected at onset of primary instability. What new features of the analysis of parts (h)-(i) enabled us to make this prediction?