Problem 1

Long wavelength instabilities of stripe state in the amplitude equation formalism. It turns out that many calculations that could, in principle, be performed using the Galerkin expansion, simplify considerably in the amplitude equation formalism. One particular example is the stability calculations for the stripe state, which can be performed using the scaled version of the amplitude equation

$$\partial_T \bar{A} = \bar{A} + (\partial_X - \frac{i}{2}\partial_Y^2)^2 \bar{A} - |\bar{A}|^2 \bar{A}. \tag{1}$$

(a) Is it possible to analyze the cross-roll instability using the amplitude equation (1)? Explain why.

(b) Find the amplitude of the nonlinear saturated stripe solution with dimensional wave number $\mathbf{q} = q\hat{\mathbf{x}}$. Express the answer in terms of the scaled wave number deviation $K = \epsilon^{-1/2}\xi_0(q - q_c)$ and scaled coordinates X.

(c) The stability of such stripe states can be tested by writing the amplitude in the form

$$\bar{A}(X, Y, T) = \bar{A}_K(X) + \delta\bar{A}(X, Y, T).$$

By linearizing (1) about $\bar{A}_K$ derive the equation satisfied by small disturbances $\delta\bar{A}$. By careful, the amplitude is complex!

(d) The solutions to the linearized equation you have derived in (c) can be found in the form of the Block perturbations, since $\bar{A}_K$ is periodic. We will need to compute the growth rate $\sigma_K(\mathbf{Q})$ of perturbations labeled by a Block wave vector $\mathbf{Q}$. The usual form of the Bloch's theorem addresses the properties of a perturbation to a real solution. To study complex solutions we need to generalize Bloch's theorem for a complex base state. The form of the generalization can be determined by trying the ansatz for the perturbation in the form $\delta\bar{A} \sim e^{iKX}e^{i\mathbf{Q}\cdot\mathbf{X}}$. Substitution into the linearized equation will give a number of terms with the same spatial dependence, but will also generate a term $e^{iKX}e^{-i\mathbf{Q}\cdot\mathbf{X}}$. Thus we make the more general ansatz

$$\delta\bar{A} \sim e^{iKX}\left[\delta a_+(t)e^{i\mathbf{Q}\cdot\mathbf{X}} + \delta a_-^*(t)e^{-i\mathbf{Q}\cdot\mathbf{X}}\right], \tag{2}$$

where we use the complex conjugate on $\delta a_-(t)$ for later convenience.

Substituting (2) into the linearized equation for $\delta\bar{A}$ and collecting the coefficients of the two independent functions $e^{\pm i\mathbf{Q}\cdot\mathbf{X}}$, show that the coefficients $\delta a_\pm(t)$ satisfy

$$\begin{aligned} \frac{d}{dt}\delta a_+ &= -(P^2 + U_+)\delta a_+ - P^2\delta a_-, \\ \frac{d}{dt}\delta a_- &= -P^2\delta a_+ - (P^2 + U_-)\delta a_-. \end{aligned} \tag{3}$$

and find $P(K)$ and $U_\pm(K, Q_X, Q_Y)$.

(e) Determine the growth rate $\sigma_K(\mathbf{Q})$ of the two solutions to the system (3) for general $\mathbf{Q}$.

We test the stability of a base state solution with wave number K by finding the maximum growth rate $\sigma_K(\mathbf{Q})$, as a function of the perturbation wave vector $\mathbf{Q}$. The state with amplitude $\bar{A}_K$ is stable if this maximum growth rate is negative. It turns out that as K is increased from zero (q moves away from q_c) where the base state is stable, the instability always occurs first for either a purely longitudinal ($\mathbf{Q} = Q_X\hat{\mathbf{X}}$) or purely transverse ($\mathbf{Q} = Q_Y\hat{\mathbf{Y}}$) perturbation. Furthermore, the instability always occurs first for a long wavelength disturbance, i.e., in the limit $Q \to 0$, so the longitudinal and transverse instabilities correspond to the Eckhaus and zigzag instabilities, respectively.

(f) Compute the growth rate $\sigma_K(\mathbf{Q})$ for a perturbation wave vector, $\mathbf{Q} = Q_Y\hat{\mathbf{Y}}$, perpendicular to the wave vector of the base state. Determine the critical values of K and Q_Y and hence determine the instability type (type-I$_s$, type-III$_o$, etc). Translate the result for K_c into the zigzag stability boundary in terms of the unscaled wave number q.

(g) Compute the growth rate $\sigma_K(\mathbf{Q})$ for a perturbation wave vector, $\mathbf{Q} = Q_X \hat{\mathbf{X}}$, parallel to the wave vector of the base state. By expanding the growth rate in powers of Q_X determine the critical values of K and Q_X and hence determine the instability type. Translate the result for K_c into the Eckhaus stability boundary in terms of the unscaled wave number q.

Problem 2

Stripe interaction in the amplitude equation formalism.

(a) Compute the function $G(\theta)$ for the modified Swift-Hohenberg equation:

$$\partial_t u = ru - (1 + \nabla^2)^2 u + \nabla \cdot [(\nabla u)^2 \nabla u].$$

(b) Based on these results determine the stability of stripes vs. the square lattice pattern and compare your results with those obtained previously using the Galerkin expansion approach.

Problem 3

Stability of a hexagonal state.

Show using the amplitude equations describing unmodulated (ideal) lattice states that the hexagon state $A_1 = A_2 = A_3 = A_H$ is stable to the decay towards stripes (e.g., $A_1 = A_S$, $A_2 = A_3 = 0$) and stripes are unstable towards hexagons for $G(\pi/3) < 1$ with the opposite result for $G(\pi/3) > 1$. Assume that the coefficient of the quadratic term vanishes, $\gamma = 0$.