Problem 1

(a) Setting $u = u_0 + \delta u$ and keeping only linear terms in the equation we find

$$\partial_t \delta u = r\delta u + s\partial_x \delta u - (1+\partial_x^2)^2 \delta u - 3u_0^2 \delta u.$$

For the particular solution $\delta u = \delta u_0 e^{iqx} e^{\sigma t}$ the growth rate is given by

$$\sigma_q = r - 3u_0^2 - (1-q^2)^2 + isq$$

Stability of the trivial base state $u_0 = 0$ is determined by the real part of σ_q,

$$\gamma_q = \mathrm{Re}(\sigma_q) = r - (1-q^2)^2,$$

which has a minimum $\gamma_q = r$ at $q = 1$, so $r_c = 0$ and $q_c = 1$. Near $(r, q) = (0, 1)$ we can write

$$\gamma_q = r - (1-q)^2(1+q)^2 \approx r - 2^2(q-q_c)^2,$$

so that $\tau = 1$ and $\xi_c = 2$. Finally, the critical frequency is

$$\omega_c = \mathrm{Im}(\sigma_q)_{q=q_c} = sq_c = s \neq 0,$$

so this is a type-I$_o$ instability (for $s \neq 0$).

(b) The perturbation

$$\delta u = \delta u_0 e^{iqx} e^{\sigma_q t} = \delta u_0 e^{iq(x+st)} e^{\gamma_q t}$$

describes a growing pattern with wave number q in a frame moving in the negative x direction with speed s, so the new term describes the drift of the pattern, with s being its phase velocity. In a moving reference frame, $y = x + st$ we should replace ∂_t with $\partial_t + s\partial_x$ and ∂_x with ∂_y, recovering the standard Swift-Hohenberg equation

$$\partial_t \delta u = r\delta u - (1+\partial_y^2)^2 \delta u - u^3.$$

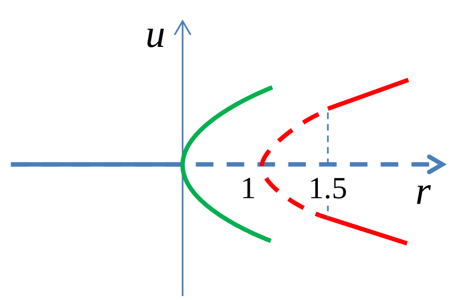
(c) Steady uniform solutions satisfy the equation

$$0 = ru_0 - u_0 + u_0^3 = (r - 1 - u_0^2)u_0,$$

so in addition to the trivial solution u_0 there are two nontrivial solutions: $u_0 = \pm\sqrt{r-1}$, which only exist for $r > 1$. Since they do not even exist at $r = 0$ when the trivial base state is destabilized, they cannot possibly be selected. This is the answer to the question.

For the curious: If you want to determine which solutions will be selected instead, this is not difficult to figure out. We already know that for $s = 0$ the uniform solution is replaced with a stationary nonuniform pattern. The same thing happens here: as the trivial base state becomes unstable, an unsteady nonuniform pattern with $q \approx 1$ (traveling wave found in part (a)) develops. It can be verified that these solutions will be stable for $r > 0$ using linear stability analysis.

The entire bifurcation diagram with the trivial uniform solution (blue), nontrivial uniform solutions (red), and the traveling wave pattern (green) is sketched in the figure below.



Problem 2

(a) To nondimensionalize the equations we introduce rescaled variables

$$a = Aa', \qquad n = Nn', \qquad t = Tt', \qquad x = Xx'.$$

In terms of the new (nondimensional) variables

$$\begin{aligned}
\frac{N}{T}\partial_t n' &= -\frac{\chi NA}{X^2}\partial_x(n'\partial_x a') + \frac{D_n N}{X^2}\partial_x^2 n', \\
\frac{A}{T}\partial_t a' &= hNn' - kAa' + \frac{D_a A}{X^2}\partial_x^2 a'.
\end{aligned}$$

Dividing each equation by the numerical prefactor on the left-hand-side (N/T or A/T) and dropping the primes we get

$$\begin{aligned}
\partial_t n &= -\frac{\chi TA}{X^2}\partial_x(n\partial_x a) + \frac{D_n T}{X^2}\partial_x^2 n, \\
\partial_t a &= \frac{hNT}{A}n - kTa + \frac{D_a T}{X^2}\partial_x^2 a.
\end{aligned}$$

We can set use four scales to set four of these prefactors to unity and the fifth prefactor to a new nondimensional parameter d:

$$\frac{\chi TA}{X^2} = 1, \qquad \frac{hNT}{A} = 1, \qquad kT = 1, \qquad \frac{D_a T}{X^2} = 1, \qquad \frac{D_n T}{X^2} = d.$$

With these choices

$$\begin{aligned}
\partial_t n &= -\partial_x(n\partial_x a) + d\partial_x^2 n, \\
\partial_t a &= n - a + \partial_x^2 a,
\end{aligned}$$

where $d = D_n/D_a$.

(b) For a steady uniform solution $a(x,t) = a_0$ and $n(x,t) = n_0$ the first equation becomes an identity, while the second one gives $a_0 = n_0$. Linearizing the only nonlinear term about $a = n = n_0$, we find

$$\partial_x(n\partial_x a) = \partial_x[(n_0+\delta n)\partial_x(n_0+\delta a)] = \partial_x(n_0+\delta n)\partial_x\delta a + (n_0+\delta n)\partial_x^2\delta a \approx n_0\partial_x^2\delta a + \cdots$$

so that

$$\begin{aligned}
\partial_t \delta n &= -\partial_x(n\partial_x a) + d\partial_x^2 n, \\
\partial_t a &= n - a + \partial_x^2 a.
\end{aligned}$$

A perturbation of the form

$$\begin{pmatrix} \delta n \\ \delta a \end{pmatrix} = \begin{pmatrix} \delta \bar{n} \\ \delta \bar{a} \end{pmatrix} e^{iqx} e^{\sigma t}$$

leads to the following eigenvalue problem

$$\begin{pmatrix} -dq^2 & n_0 q^2 \\ 1 & -1-q^2 \end{pmatrix} \begin{pmatrix} \delta \bar{n} \\ \delta \bar{a} \end{pmatrix} \equiv J \begin{pmatrix} \delta \bar{n} \\ \delta \bar{a} \end{pmatrix} = \sigma \begin{pmatrix} \delta \bar{n} \\ \delta \bar{a} \end{pmatrix}$$

Although we could compute the two eigenvalues $\sigma_\pm$ explicitly, there is no reason to do this, since we can determine the stability of the uniform state entirely in terms of

$$\Delta = \det(J) = \sigma_-\sigma_+ = (d(q^2+1) - n_0)q^2$$

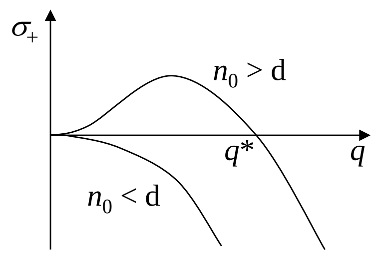
and

$$\tau = \mathrm{tr}(J) = \sigma_- + \sigma_+ = -(d+1)q^2 - 1.$$

Since $\tau < 0$, the two eigenvalues cannot be complex conjugate when the stability boundary is crossed. For two real eigenvalues stability boundary is determined by one of the eigenvalues vanishing, when $\Delta = 0$ and we should either have $q = 0$ or (for $n_0 > d$) $q = q^*$, where

$$q^* = \sqrt{\frac{n_0}{d} - 1}.$$

Let $\sigma_- \leq \sigma_+$, then when $\sigma_+ = 0$, we have $\sigma_- = \tau < 0$. Therefore, the largest eigenvalue is always zero at $q = 0$ and this is a type-II$_s$ instability. For this instability the equation $\sigma_+ = 0$ has only one root when the base state is stable and two when it is unstable, hence the growth rate looks like the sketch below.



(c) In a finite system with no-flux boundary conditions only discrete wave numbers $q_n = \pi n/L$ are possible, where L is the system size. In order for the instability to occur we need at least one of these modes to lie in the unstable band, $0 < q_n < q^*$. The mode $q_0 = 0$ does not describe a pattern. As L increases, the $n = 1$ mode becomes unstable first at

$$L = X\frac{\pi}{q^*} = \pi\left(\frac{D_a}{\chi}\right)^{1/2}\left(\frac{n_0}{d} - 1\right)^{-1/2} \equiv L_1.$$

At the onset of the instability this length clearly diverges, as one would expect.

(d) The increase in χ leads to the increase in the ratio n_0/d, if all other parameters are held fixed. Indeed, if we denote the dimensional concentration of amoebae N_0, then

$$\frac{n_0}{d} = \frac{N_0}{Nd} = \frac{N_0 D_a \chi h}{D_n D_a k} = \frac{N_0 h \chi}{D_n k} \equiv \frac{\chi}{\chi_0}.$$

In an infinite system, as n_0 is increased past d (or χ exceeds χ_0), the Fourier modes with $q \approx 0$ get destabilized first, in other words, the critical wavelength is infinite. If alternatively n_0 and d are fixed and the size L is gradually increased, the mode with $q = q^*$ is destabilized first, so the corresponding wavelength is equal to $2L_1$. In either case it is of order the size of the system, though.

(e) According to the effect of different terms on the dynamics of concentrations, the meaning of parameters is as follows:

χ - sensitivity of amoebae to the attractant (how fast they "swim" in the direction of increasing concentration of C-AMP),

D_n - amoeba diffusion constant,

D_a - attractant (C-AMP) diffusion constant,

k - the degradation rate of the attractant, and

h - the rate at which amoebae produce the attractant. The term represents negative diffusion resulting in depletion of regions of low concentration of amoebas and further concentration where their density is large.

When $\chi < \chi_0$ (n is too low), dissipation (diffusion and degradation of C-AMP) maintain the uniform state. The nonlinear term $-\chi\partial_x(n\partial_x a) = \partial_x j_n$ describes a change in the density of amoeba due to the flux $j_n = -\chi n \partial_x a$ of amoebas in the direction of increasing C-AMP concentration, where the product $-\chi n$ serves as an effective negative(!) diffusion constant. This flux is enhanced as χ (or n) increase above threshold, overwhelming dissipation and leading to an instability causing spatial agregation. Spatial aggregation further increases production of C-AMP in the regions of high concentration of amoebas, increasing the gradient of C-AMP and speeding up the aggregation process.