Problem 1

(a) The dimensions v, L, and ν are $[v] = LT^{-1}$, $[L] = L$, $[\nu] = L^2T^{-1}$, so the dimension of the Reynolds number R is:

$$[R] = \frac{[v][L]}{[T]} = \frac{LT^{-1} \cdot L}{L^2T^{-1}} = 1,$$

i.e., R is dimensionless.

(b) The speed of the airflow over the wing is $v = 500$ km/h, the characteristic length scale (wing chord or width) is $L \approx 2.5$ m, and the viscosity of air is $\nu = 1.5 \times 10^{-5}$ m^2/s. The corresponding Reyndolds number is

$$R = \frac{vL}{\nu} = \frac{500 \text{ km/h} \cdot 2.5 \text{ m}}{1.5 \times 10^{-5} \text{ m}^2/\text{s}} \cdot \frac{1000 \text{ m/km}}{3600 \text{ s/h}} = 2.31 \times 10^7,$$

which is much larger than $R_c = 1000$, i.e., the flow is highly turbulent.

(c) For the airflow around human foot we estimate $v \approx 1$ m/s, $L \approx 0.3$ m, and $\nu = 1.5 \times 10^{-5}$ m^2/s. The corresponding Reyndolds number is

$$R = \frac{vL}{\nu} = \frac{1 \text{ m/s} \cdot 0.3 \text{ m}}{1.5 \times 10^{-5} \text{ m}^2/\text{s}} = 2 \times 10^4,$$

which is larger than $R_c = 1000$, i.e., the flow is turbulent.

(d) Flipping a coin imparts it a velocity of order $v = 3$ m/s and rotational frequency of order $f = 20$ Hz. The radius of a coin is about $r = 0.01$ m (so $L = 2r = 0.02$ m). The velocity of the edges relative to the center of mass is $2\pi fr = 1.2$ m/s, comparable to v, so rotation would not change the Reynolds number significantly. We therefore find

$$R = \frac{vL}{\nu} = \frac{3 \text{ m/s} \cdot 0.02 \text{ m}}{1.5 \times 10^{-5} \text{ m}^2/\text{s}} = 4 \times 10^3,$$

which is larger than $R_c = 1000$, i.e., the flow is weakly turbulent and can affect the outcome of a coin toss.

(e) The blood is ejected by the ventricles, each of which makes up about a quarter of the total volume of the heart. The heart is about $12 \times 8 \times 6$ cm in size, so the volume of a ventricle is about

$$V_0 \approx \frac{12 \text{ cm} \cdot 8 \text{ cm} \cdot 6 \text{ cm}}{4} = 1.5 \times 10^{-4} \text{ m}^3.$$

We can assume that about 50% of this volume of blood is ejected per contraction, i.e., the blood volume ejected is $V \approx 7 \times 10^{-5}$ m^3. The heart contracts about 75 times per minute, i.e., with frequency $f \approx 1.25$ Hz. The radius of the heart valve is $r = 1$ cm (about the same as the radius of the aorta), so its the cross-section is

$$A \approx \pi r^2 \approx 2 \times 10^{-4} \text{ m}^2,$$

and the velocity of the blood through the valve is

$$v \approx \frac{Vf}{A} \approx 0.3 \text{ m/s}.$$

The corresponding Reynolds number is

$$R = \frac{v \cdot 2r}{\nu} = \frac{0.3 \text{ m/s} \cdot 0.02 \text{ m}}{3 \times 10^{-6} \text{ m}^2/\text{s}} = 2000,$$

which is larger than $R_c = 1000$, i.e., the flow is weakly turbulent.

The arteries are a few millimeters wide, so we can take $r \approx 2$ mm. The velocity of the blood flow through the arteries is comparable to that through the aorta (and the heart valve), since the total cross-sectional area of the arteries is about the same as that of the aorta. Hence, in the arteries

$$R = \frac{v \cdot 2r}{\nu} = \frac{0.3 \text{ m/s} \cdot 0.004 \text{ m}}{3 \times 10^{-6} \text{ m}^2/\text{s}} = 400,$$

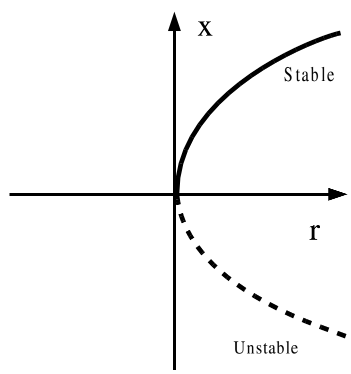
which is less than $R_c = 1000$, i.e., the flow is laminar.

(f) For the sound to be generated, the air flow must be turbulent (or at least unsteady), so

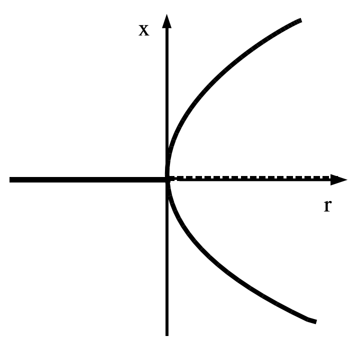
$$v > \frac{R_c \nu}{L} = \frac{1000 \cdot 1.5 \times 10^{-5} \text{ m}^2/\text{s}}{0.002 \text{ m}} = 7.5 \text{ m/s}.$$

Problem 2

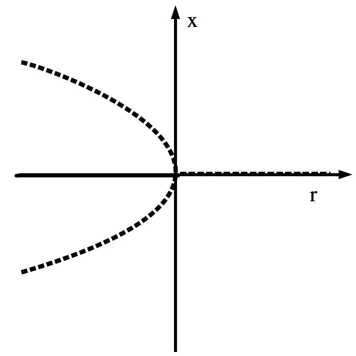
(a) Saddle-node bifurcation $\dot{x} = f(x) = r - x^2$. The fixed points are $x^* = \pm\sqrt{r}$ (only for $r \geq 0$). Stability requires $f'(x^*) < 0$. Since $f'(x) = -2x$, $x^* = \sqrt{r}$ is stable and $x^* = -\sqrt{r}$ is unstable.



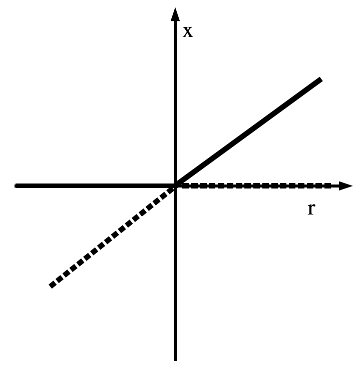
(b) Supercritical pitchfork bifurcation $\dot{x} = f(x) = rx - x^3$. The fixed points are $x^* = 0$ (for all r) and $\pm\sqrt{r}$ (for $r > 0$). We have $f'(x) = r - 3x^2$, so $x^* = 0$ is stable for $r < 0$, unstable for $r > 0$. The fixed points $x^* = \pm\sqrt{r}$ are both stable, since $f'(x^*) = -2r < 0$ for both.



(c) Subcritical pitchfork bifurcation $x = rx + x^3$. The fixed points are $x^* = 0$ (for all r) and $\pm\sqrt{-r}$ (for $r < 0$). We have $f'(x) = r + 3x^2$, so $x^* = 0$ is stable for $r < 0$, unstable for $r > 0$. Fixed points $x^* = \pm\sqrt{-r}$ are unstable, since $f'(x^*) = -2r > 0$ for both.



(d) Transcritical bifurcation $x = rx - x^2$. The fixed points are $x^* = 0, r$ for all r. We have $f'(x) = r - 2x$, so $x^* = 0$ is stable for $r < 0$, unstable for $r > 0$. For $x^* = r$ we have $f'(x^*) = -r$, so this fixed point is unstable for $r < 0$, stable for $r > 0$.



Problem 3

(a) Since $u(x) = u_0$, we have $u_x = 0$ and

$$\begin{aligned} F[u_0 + h] - F[u_0] &= (u_0 + h)(u_0 + h)_x - u_0(u_0)_x \\ &= uh_x + hh_x \\ &\equiv J[h] + O(h^2), \end{aligned} \tag{1}$$

so that

$$J = u_0 \partial_x. \tag{2}$$

(b) Since $u(x) = u_0$, we have $\nabla u = 0$ and

$$\begin{aligned} F[u_0 + h] - F[u_0] &= \nabla^2 (u_0 + h)^2 - \nabla^2 u_0^2 \\ &= \nabla \cdot \nabla (u_0^2 + 2u_0 h + h^2) \\ &= \nabla \cdot (2u_0 \nabla h + 2h \nabla u_0 + \nabla h^2) \\ &= 2u_0 \nabla^2 h + \nabla^2 h^2 \\ &\equiv J[h] + O(h^2) \end{aligned} \tag{3}$$

so that

$$J = 2u_0 \nabla^2. \tag{4}$$

(c) Since $\mathbf{u}_0(x) = \mathbf{u}_0$, we have $\partial_i u_j = 0$ for $i, j = 1, 2, 3$ and

$$\begin{aligned} F[\mathbf{u}_0 + \mathbf{h}] - F[\mathbf{u}_0] &= (\mathbf{u}_0 + \mathbf{h}) \cdot \nabla(\mathbf{u}_0 + \mathbf{h}) - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 \\ &= (\mathbf{u}_0 + \mathbf{h}) \cdot \nabla \mathbf{h} \\ &= \mathbf{u}_0 \cdot \nabla \mathbf{h} + \mathbf{h} \cdot \nabla \mathbf{h} \\ &\equiv J[\mathbf{h}] + O(h^2) \end{aligned} \tag{5}$$

so that

$$J = \mathbf{u}_0 \cdot \nabla. \tag{6}$$