Solutions for Assignment #10

Note Title 4/27/2006

Problem 1 We know that a time-independent amplitude equation

$$0 = \bar{A} + \partial_x^2 \bar{A} - \bar{A}^3 \quad \text{has solutions} \quad \bar{A} = 0,\ 1,\ \tanh\left(\frac{x}{\sqrt{2}}\right).$$

The front solution for the time-dependent version

$$\partial_T \bar{A} = \bar{A} + \partial_x^2 \bar{A} - \bar{A}^3$$

has to connect states $\bar{A} = 1$ (nonlinearly saturated pattern) for $x \to \infty$ with $\bar{A} = 0$ (uniform state) for $x \to -\infty$, so pick a trial solution in the form:

$$\bar{A}(x,t) = \frac{1}{2}\left(1 + \tanh k(x - ct)\right)$$

In the comoving reference frame

$$\partial_\xi^2 \bar{A} + \partial_\xi \bar{A} + \bar{A} - \bar{A}^3 = 0 \quad \text{and} \quad \bar{A} = \frac{1}{2}\left(1 + \tanh k\xi\right).$$

After some algebra we find that the equation is satisfied by picking $k = \frac{1}{2\sqrt{2}}$ and $c = \frac{3}{\sqrt{2}}$, that is

$$\bar{A}(x,t) = \frac{1}{2}\left(1+\tanh\left(\frac{1}{2\sqrt{2}}\left(x-\frac{3}{\sqrt{2}}t\right)\right)\right)$$

Going to the unscaled variables we obtain

$$A(x,t) = \frac{1}{2}\sqrt{\frac{\varepsilon}{g_0}}\left(1+\tanh\left[\frac{\sqrt{\varepsilon}\,x}{2\sqrt{2}\,\xi_0} - \frac{3\varepsilon t}{4\tau_0}\right]\right)$$

Problem 2

(a) The expression for the growth rate of disturbances about the uniform state $u=0$ is

$$\sigma(q) = r-(1-q^2)^2$$

The velocity of the pulled front satisfies the equations

$$c = \frac{\mathrm{Re}\,\sigma(q_s)}{\mathrm{Im}(q_s)} = i\left.\frac{d\sigma}{dq}\right|_{q=q_s} = 4iq_s(1-q_s^2)$$

Substituting $q_s = u+iv$ and evaluating the real and imaginary parts of the equations, we find the following real equations:

$$c + 4v - 12u^3 v + 4v^3 = 0$$

$$4u(1 - u^2 + 3v^2) = 0$$

$$cv - ru + 1 - 2u^2 + 2v^2 + u^4 - 6u^2v^2 + v^4 = 0$$

The solution with real u, v, and c is

$$u = \pm \frac{1}{2}\sqrt{3 + \sqrt{1+6r}}, \quad v = \pm \frac{1}{6}\sqrt{-3 + 3\sqrt{1+6r}},$$

$$c = \pm \frac{4}{3}\left(\frac{2}{3} + \frac{1}{3}\sqrt{1+6r}\right)\cdot\sqrt{-3+3\sqrt{1+6r}},$$

which agree with those given in the problem statement.

(b) For small r, we find (for the positive solutions):

$$q = \left(1 + \frac{3}{8}r + \dots\right) + i\left(\frac{1}{2}\sqrt{r} + \dots\right) \approx 1$$

and $\quad c = 4\sqrt{r} + \dots$

Indeed for SH equation we have $q_c = 1$, $\xi_0 = 2$, and $\tau_0 = 1$.

In Problem 1 we found

$$c = \frac{3}{2}\sqrt{2\varepsilon}\,\frac{\xi_0}{\tau_0} = 3\sqrt{2}\sqrt{\varepsilon} \overset{\varepsilon = r}{=} 3\sqrt{2}\sqrt{r} \neq 4\sqrt{r}$$

This has the same scaling, but a different prefactor, which puts the validity of the pulled front assumption in question.

(c) The wavelength is given by

$$q_{fp} = \mathrm{Re}(q_s) + \frac{1}{c}\mathrm{Im}(G(q_s)) =$$

$$= \frac{1}{c}\left(uc + r - 1 + 2u^2 - 2v^2 - u^4 + 6u^2v^2 - v^4\right) =$$

$$= \cdots = \frac{3\sqrt{3+\sqrt{1+6r}}}{8\left(2+\sqrt{1+6r}\right)}$$

Problem 3

(a) Moving into the reference frame, $u = u(\xi)$, $\xi = x - ct$ we find

$$u'' + cu' + F(u) = 0,$$

which can be rewritten as $\begin{cases} u' = v, \\ v' = -cv - F(u). \end{cases}$

(b) The fixed points should have $u' = v = 0$ and $v' = -cv - F(u) = 0$, which requires $F(u) = 0$. Altogether, there are five real solutions for $-\frac{1}{4} < \varepsilon < 0$:

$$u = 0, \quad u = \pm\frac{1}{2}\sqrt{2 \pm 2\sqrt{1+4\varepsilon}}$$

The growth rate of disturbances about the uniform state with

<u>$u = 0$:</u> $\quad \sigma = \varepsilon - q^2 \Rightarrow$ stable for $\varepsilon < 0$

unstable for $\varepsilon > 0$

<u>$u = \pm\frac{1}{2}\sqrt{2+2\sqrt{1+4\varepsilon}}$:</u> $\quad \sigma = -1 - 4\varepsilon - \sqrt{1+4\varepsilon} - q^2$

$\Rightarrow$ stable for $\varepsilon > -1/4$

<u>$u = \pm\frac{1}{2}\sqrt{2-2\sqrt{1+4\varepsilon}}$:</u> $\quad \sigma = -1 - 4\varepsilon + \sqrt{1+4\varepsilon} - q^2$

$\Rightarrow$ unstable for $-\frac{1}{4} < \varepsilon < 0$.

Summing up, only the solution $(u,v) = \left(\pm\frac{1}{2}\sqrt{2+2\sqrt{1+4\varepsilon}}, 0\right)$ is stable for all $\varepsilon > -\frac{1}{4}$.

(c) The Jacobian of the system $v' = u$, $u' = -cv - F(u)$ is

$$J = \begin{pmatrix} 0 & 1 \\ -F'(u) & -c \end{pmatrix}$$

Since $\mathrm{Tr}(J) = \lambda_1 + \lambda_2 = -c < 0$

and $\det(J) = \lambda_1 \cdot \lambda_2 = F'(u) = -1 - 4\varepsilon - \sqrt{1+4\varepsilon} < 0$

this fixed point is always a saddle with one positive and one negative eigenvalue / direction for any $c > 0$.

(d) For $(u,v) = (0,0)$ the eigenvalues are

$$\lambda_{1,2} = -\frac{1}{2}c \pm \frac{1}{2}\sqrt{c^2 - 4\varepsilon}$$

and we find the following possibilities

$\underline{|c| < 2\sqrt{\varepsilon}}$: $\lambda_{1,2} = -\frac{1}{2}c \pm i\omega$ => stable spiral for $c>0$,

unstable spiral for $c<0$.

$\underline{|c| > 2\sqrt{\varepsilon}}$: $\lambda_1 = -\frac{1}{2}c + \frac{1}{2}\sqrt{c^2-4\varepsilon} < 0$, $c>0$

> 0, $c<0$

$\lambda_2 = -\frac{1}{2}c - \frac{1}{2}\sqrt{c^2-4\varepsilon} < 0$, $c>0$

> 0, $c<0$

=> stable node for $c>0$,

unstable node for $c<0$.

(e)