Problem 1

Linearizing the evolution equation about the base state $u_0 = 0$ we find

$$\partial_t^2 \delta u = r\delta u - (1+\nabla^2)^2 \delta u.$$

Substituting $\delta u_{\mathbf{q}} = Ae^{i\mathbf{q}\cdot\mathbf{x}+\sigma_{\mathbf{q}}t}$ we find

$$\sigma_{\mathbf{q}}^2 = r - (1-q^2)^2.$$

For $r < 0$ we have $\sigma_{\mathbf{q}}^2 < 0$, so that $\mathrm{Re}(\sigma_{\mathbf{q}}) = 0$ and the base state is stable. For $r > 0$ we have $\mathrm{Re}(\sigma_{\mathbf{q}}) > 0$ for $1-\sqrt{r} < q^2 < 1+\sqrt{r}$, i.e., the base state is unstable and the critical parameter is $r_c = 0$. The maximal growth rate is achieved at $q = q_c = 1$. At $r = r_c = 0$ and $q = q_c$ we have $\mathrm{Re}(\sigma_{\mathbf{q}}) = \mathrm{Im}(\sigma_{\mathbf{q}}) = 0$, so that $\omega_c = 0$ and the instability is of type I_s.

Problem 2

Linearizing the equations about the base state $(u_0, v_0) = (0,0)$ we obtain

$$\partial_t \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = J \begin{pmatrix} \delta u \\ \delta v \end{pmatrix},$$

where the Jacobian is

$$J = \begin{pmatrix} a+\partial_x^2 & -b\partial_x^2 \\ b\partial_x^2 & a+\partial_x^2 \end{pmatrix}.$$

Substituting $\delta u_q = Ue^{iqx+\sigma_q t}$ and $\delta v_q = Ve^{iqx+\sigma_q t}$ we obtain an eigenvalue problem

$$\begin{pmatrix} a-q^2 & bq^2 \\ -bq^2 & a-q^2 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \sigma_q \begin{pmatrix} U \\ V \end{pmatrix}.$$

The eigenvalues are

$$\sigma_q = a - q^2 \pm ibq^2.$$

The maximum growth rate $\mathrm{Re}(\sigma_q) = a$ is achieved at $q = q_c = 0$, so for $a < 0$ the uniform state is stable. For $a > 0$ we have $\mathrm{Re}(\sigma_q) > 0$ for $q^2 < a$ and the uniform state is unstable. Therefore, the critical parameter value is $a_c = 0$ and the critical wave number is $q_c = 0$. The critical frequency $\omega_c = \mathrm{Im}(\sigma_{q_c}) = 0$, so the instability is of type III_s.

Problem 3

(a) Linearizing about the Swift-Hohenberg equation about the uniform state u we find

$$\partial_t \delta u = \left(r - (1+\partial_x^2)^2\right)\delta u - 3u^2\delta u. \tag{1}$$

Substituting $u = 0$ yields

$$\partial_t \delta u = \left(r - (1+\partial_x^2)^2\right)\delta u. \tag{2}$$

(b) Since $u = u' = 0$ and $u + \delta u = (u+\delta u)' = 0$ (we are using the shorthand notation $u' = \partial_x u$) at $x = \pm L/2$, we must also have $\delta u = \delta u' = 0$ at $x = \pm L/2$.

(c) Whether $f = \cos(qx)$ or $f = \sin(qx)$, we have $f'' = -q^2 f$, so that

$$(1+\partial_x^2)^2 f = f + 2\partial_x^2 f + \partial_x^4 f = f - q^2 f - q^2\partial_x^2 f = f - q^2 f + q^4 f = (1-q^2)^2 f.$$

Hence $rf = (1-q^2)^2 f$ and $r = (1-q^2)^2$. Solving this for q we find

$$q_\pm = \sqrt{1 \pm \sqrt{r}}.$$

(d) Applying the boundary conditions at $x = L/2$ (boundary conditions at $x = -L/2$ will be satisfied automatically) to

$$f(x) = A_+ \cos(q_+ x) + A_- \cos(q_- x) \tag{3}$$

we find

$$A_+ \cos(q_+ L/2) + A_- \cos(q_- L/2) = 0 \tag{4}$$
$$A_+ q_+ \sin(q_+ L/2) + A_- q_- \sin(q_- L/2) = 0, \tag{5}$$

so that

$$\frac{A_+}{A_-} = \frac{\cos(q_- L/2)}{\cos(q_+ L/2)} \tag{6}$$

and

$$\frac{q_+}{q_-} = -\frac{A_- \sin(q_- L/2)}{A_+ \sin(q_+ L/2)} = \frac{\tan(q_- L/2)}{\tan(q_+ L/2)} \tag{7}$$

or, in terms of r,

$$g_e(r) = \sqrt{1+\sqrt{r}}\tan\left(\sqrt{1+\sqrt{r}}\frac{L}{2}\right) - \sqrt{1-\sqrt{r}}\tan\left(\sqrt{1-\sqrt{r}}\frac{L}{2}\right) = 0. \tag{8}$$

Repeating these calculations for

$$f = A_+ \sin(q_+ x) + A_- \sin(q_- x) \tag{9}$$

we obtain (skipping the details)

$$g_o(r) = \sqrt{1+\sqrt{r}}\cot\left(\sqrt{1+\sqrt{r}}\frac{L}{2}\right) - \sqrt{1-\sqrt{r}}\cot\left(\sqrt{1-\sqrt{r}}\frac{L}{2}\right) = 0. \tag{10}$$

(e) Since in the unbounded system the critical parameter is $r_c = 0$, for the large, but bounded, system we should have $0 < r_c \ll 1$ (boundary conditions delay the onset of the bifurcation slightly). Hence, to leading order

$$\tan\left[\left(1-\frac{\sqrt{r}}{2}\right)\frac{L}{2}\right] - \tan\left[\left(1+\frac{\sqrt{r}}{2}\right)\frac{L}{2}\right] = 0$$

or

$$\left[\left(1+\frac{\sqrt{r}}{2}\right)\frac{L}{2}\right] = \left[\left(1-\frac{\sqrt{r}}{2}\right)\frac{L}{2}\right] + \pi n$$

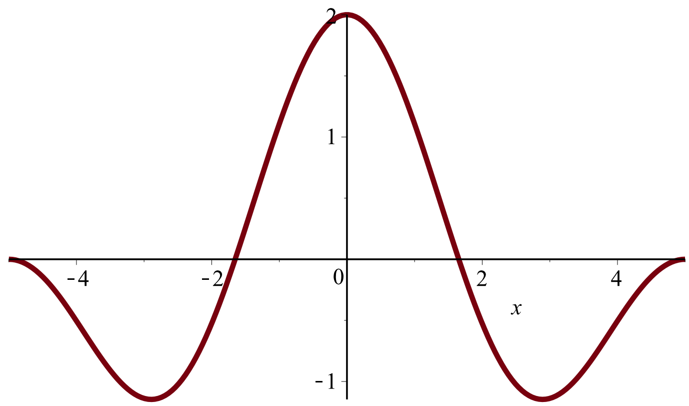
with some integer n. The smallest value of $\sqrt{r}$ corresponds to $n = 1$ and gives

$$\sqrt{r_c} \approx \frac{2\pi}{L} \tag{11}$$

giving the desired expression.

(f) For $L = 10$ we have $r_c^a = 0.394784$ (where subscript stands for "analytic" or "approximate"). For $f(x)$-even, $g_e(r) = 0$ yields $r_c^e = 0.350724 < r_c^a$. Evaluating $A_\pm$ and $q_\pm$ we find

$$f(x) = \cos(1.261831x) - 1.000979\cos(0.638576x).$$



For $f(x)$-odd, $g_o(r) = 0$ yields $r_c^o = 0.359998 > r_c^a$. Evaluating $A_\pm$ and $q_\pm$ we find

$$f(x) = \sin(1.264910x) + 1.998776\sin(0.632456x).$$