Problem 1

(a) The eigenvalues of a 2×2 real matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

satisfy the characteristic equation

$$(a_{11} - \sigma)(a_{22} - \sigma) - a_{12}a_{21} = \sigma^2 - \text{tr}(A)\sigma + \det(A) = 0,$$

where $\det(A) = a_{11}a_{22} - a_{12}a_{21}$ and $\text{tr}(A) = a_{11} + a_{22}$, and therefore can be written in the form

$$\sigma_\pm = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}.$$

When $\tau^2 - 4\delta \geq 0$, the eigenvalues are real. If $\delta > 0$, the sign of τ determines the sign of both eigenvalues. If $\delta < 0$, the eigenvalues have opposite signs. When $\tau^2 - 4\delta < 0$, the eigenvalues are complex conjugate. In this case $\text{Re}(\sigma_\pm) = \tau/2 < 0$ and $\delta > \tau^2/4 > 0$. Therefore, in both cases, $\text{Re}(\sigma_\pm) < 0$ if and only if $\tau < 0$ and $\delta > 0$.

Alternatively, when $\sigma_\pm$ are both real, $\sigma_\pm < 0$ if and only if $\sigma_- + \sigma_+ = \tau < 0$ and $\sigma_-\sigma_+ = \delta > 0$. When the eigenvalues are complex conjugate, $\sigma_\pm = \gamma \pm i\omega$, we have $\delta = \sigma_-\sigma_+ = \gamma^2 + \omega^2 > 0$ (always) and $\tau = 2\gamma < 0$ if and only if $\text{Re}(\sigma_\pm) < 0$.

(b) For any A we can find a coordinate transformation S such that diagonalizes A, i.e.,

$$\Sigma = S^{-1}AS = \begin{pmatrix} \sigma_- & 0 \\ 0 & \sigma_+ \end{pmatrix}$$

or, in the rare case when $\sigma_- = \sigma_+ = \sigma$ and A has only one eigenvector, converts it to a Jordan normal form

$$\Sigma = S^{-1}AS = \begin{pmatrix} \sigma & 1 \\ 0 & \sigma \end{pmatrix}.$$

Let us define $\mathbf{v} = S\mathbf{u}$, such that $\dot{\mathbf{v}} = S\dot{\mathbf{u}} = SA\mathbf{u} = SAS^{-1}S\mathbf{u} = \Sigma\mathbf{v}$. In the former case (when A is diagonalizable), the two variables are uncoupled, so $v_1(t) = v_1(0)e^{-\sigma_- t}$ and $v_2(t) = v_2(0)e^{-\sigma_+ t}$. Since $\text{Re}(\sigma_\pm) < 0$, for $t \to \infty$ we have $\mathbf{v} \to 0$ and therefore $\mathbf{u} = S^{-1}\mathbf{v} \to 0$.

In the latter case, the equations for v_1 and v_2 are coupled. Solving them sequentially, we find $v_2(t) = v_2(0)e^{-\sigma t}$ and $v_1(t) = (v_1(0) + v_2(0)t)e^{-\sigma t}$. In this case too we have $\mathbf{v} \to 0$ and therefore $\mathbf{u} = S^{-1}\mathbf{v} \to 0$ for $\sigma < 0$.

Problem 2

(a) The linearization about the stationary base state $u_{10} = a$, $u_{20} = b/a$ gives for $\delta u_1 = u_1 - u_{10}$ and $\delta u_2 = u_2 - u_{20}$

$$\partial_t \delta u_1 = -(b+1)\delta u_1 + u_{10}^2 \delta u_2 + 2u_{10}u_{20}\delta u_1 + D_1 \partial_x^2 \delta u_1 = (b-1)\delta u_1 - a^2 \delta_u + D_1 \partial_x^2 \delta u_1$$

and

$$\partial_t \delta u_2 = b\delta u_1 - u_{10}^2 \delta u_2 - 2u_{10}u_{20}\delta u_1 + D_2 \partial_x^2 \delta u_2 = -b\delta u_1 - a^2 \delta u_2 + D_2 \partial_x^2 \delta u_2$$

or, in matrix-vector notation,

$$\partial_t \begin{pmatrix} \delta u_1 \\ \delta u_2 \end{pmatrix} = \begin{pmatrix} b - 1 - D_1 q^2 & a^2 \\ -b & -a^2 - D_2 q^2 \end{pmatrix} \begin{pmatrix} \delta u_1 \\ \delta u_2 \end{pmatrix} = A_q \begin{pmatrix} \delta u_1 \\ \delta u_2 \end{pmatrix}$$

(b) The eigenvalues of the Jacobian A_q are given by

$$\sigma_q^{\pm} = \frac{\tau_q \pm \sqrt{\tau_q - 4\delta_q}}{2},$$

where

$$\tau_q = \mathrm{tr}(A_q) = b - 1 - a^2 - (D_1 + D_2)q^2,$$
$$\delta_q = \det(A_q) = a^2 + [a^2 D_1 + (1-b)D_2]q^2 + D_1 D_2 q^4.$$

The Turing instability corresponds to σ_q^+ changing sign, which happens at

$$b_c = \left(1 + \sqrt{\frac{D_1}{D_2}}a\right)^2 = 2.342$$

for the parameters given. The critical wavenumber is

$$q_c = q_m = \sqrt{\frac{(b-1)D_2 - a^2 D_1}{2D_1 D_2}} = 0.435.$$

Let $b = b_c(1+\varepsilon)$. Taylor-expanding σ_q^+ around $b = b_c$ and $q = q_c$ (use Maple or Mathematica to simplify the calculations) we find

$$\sigma_q^+ = 1.749\varepsilon - 8.364(q - q_c)^2 + \cdots = 1.749[\varepsilon - 4.782(q - q_c)^2] + \cdots$$

Comparing this with the general expression $\sigma_q^+ \approx \tau^{-1}[\varepsilon - \xi_c^2(q - q_c)^2]$, we find $\tau = 0.572$ and $\xi_c = 2.186$. (If you defined $\varepsilon = b - b_c$, you should get $\tau = 1.334$ and $\xi_c = 3.347$.) This coherence length is, in fact, smaller than the wavelength λ, so we should not expect to see a particularly regular pattern for $\epsilon = O(1)$.

To verify the coherence length we can compute the solution of the evolution equations on a domain with arbitrary boundary conditions and length $L \gg \lambda$ for $\epsilon \ll 1$. The Fourier spectrum of the solution should only contain modes in the range $q_c - \varepsilon^{1/2}\xi_c^{-1} < q < q_c + \varepsilon^{1/2}\xi_c^{-1}$. Alternatively, we could measure how quickly, for Dirichlet boundary conditions, the ampitude of the pattern grows to its asymptotic value far from the boundaries.

(c) We already know that the oscillatory instability occurs for $b = b_o = 1 + a^2$. In order for the oscillatory instability to not be preempted by the Turing instability we should have $b_o < b_c$ or, equivalently,

$$1 + a^2 < 1 + 2\sqrt{\frac{D_1}{D_2}}a + \frac{D_1}{D_2}a^2.$$

Solving this for a we find

$$a < \frac{2\sqrt{D_1 D_2}}{D_2 - D_1} = 0.808$$

for the particular parameters given.

Problem 3

(a) The reacton rates are

$$r_1 = k_1[BrO_3^-][Br][H^+]^2 = K_1 w$$
$$r_2 = k_2[HBrO_2][Br][H^+] = K_2 uw$$
$$r_3 = k_3[BrO_3^-][HBrO_2][H^+]^3[Ce^{3+}]^2 = K_3 u$$
$$r_4 = k_4[HBrO_2]^2 u^2 = K_4 u^2$$
$$r_5 = k_5[Z][Ce^{4+}][H_2O]^{\alpha(h)} = K_5 v$$

so that

$$\dot{u} = r_1 - r_2 - r_3 + 2r_3 - 2r_4 = K_1 w - K_2 uw + K_3 u - 2K_4 u^2$$
$$\dot{v} = 2r_3 - r_5 = 2K_3 u - K_5 v$$
$$\dot{w} = -r_1 - r_2 + hr_5 = -K_1 w - K_2 uw + hK_5 v$$

(b) Let us introduce nondimensional variables (denoted with a bar) via

$$u = U\bar{u}, \qquad v = V\bar{v}, \qquad w = W\bar{w}, \qquad t = T\bar{t}.$$

In the new variables the evolution equations become (we use prime to denote the derivate with respect to $\bar{t}$)

$$\begin{aligned}
&UT^{-1}\bar{u}' = K_1 W\bar{w} - K_2 UW\bar{u}\bar{w} + K_3 U\bar{u} - 2K_4 U^2\bar{u}^2 \\
&VT^{-1}\bar{v}' = 2K_3 U\bar{u} - K_5 V\bar{v} \\
&WT^{-1}\bar{w}' = -K_1 W\bar{w} - K_2 UW\bar{u}\bar{w} + hK_5 V\bar{v}
\end{aligned}$$

or

$$\begin{aligned}
&(K_3T)^{-1}\bar{u}' = K_1 W(UK_3)^{-1}\bar{w} - K_2 WK_3^{-1}\bar{u}\bar{w} + \bar{u} - 2K_4 UK_3^{-1}\bar{u}^2 \\
&\bar{v}' = 2K_3 UTV^{-1}\bar{u} - K_5 T\bar{v} \\
&(TK_2U)^{-1}\bar{w}' = -K_1(K_2U)^{-1}\bar{w} - \bar{u}\bar{w} + hK_5 V(K_2UW)^{-1}\bar{v}
\end{aligned}$$

Converting the first equation into the desired form requires

$$\eta = (K_3T)^{-1}, \qquad K_1W(UK_3)^{-1} = q, \qquad K_2WK_3^{-1} = 1, \qquad 2K_4UK_3^{-1} = 1.$$

Converting the second equation into the desired form requires

$$K_5T = 1, \qquad 2K_3UTV^{-1} = 1.$$

Finally, converting the third equation into the desired form requires

$$(TK_2U)^{-1} = \eta', \qquad K_1(K_2U)^{-1} = q, \qquad hK_5V(K_2UW)^{-1} = 1.$$

Solving this system of equations yields the scales

$$T = \frac{1}{K_5}, \qquad U = \frac{K_3}{2K_4}, \qquad V = \frac{k_3^2}{K_4K_5}, \qquad W = \frac{K_3}{K_2}$$

and new parameters

$$\eta = \frac{K_5}{K_3}, \qquad q = \frac{2K_1K_4}{K_2K_3}, \qquad \eta' = \frac{2K_4K_5}{K_2K_3}, \qquad b = 2h.$$

With these definitions (and also dropping the bars and replacing primes with dots for the time differentiation), we obtain

$$\begin{aligned}
&\eta\dot{u} = qw - uw + u - u^2 \\
&\dot{v} = u - v \\
&\eta'\dot{w} = -qw - uw + bv
\end{aligned}$$

(c) For $\eta' \ll \eta \ll 1$ we can set $-qw - uw + bv = 0$. Solving this for w and substituting into the equation for $\dot{u}$ we obtain

$$\begin{aligned}
&\eta\dot{u} = u - u^2 - bv\tfrac{u-q}{u+q} \\
&\dot{v} = u - v
\end{aligned}$$