Problem 1

(a) In terms of the new time $t = c_t\bar{t}$ the evolution equation takes the form

$$\frac{m}{c_t^2}\theta'' + \frac{\alpha}{c_t}\theta' + C\sin(\theta) = A\sin(\omega_0 t),$$

where we defined $\theta' = d\theta/dt'$. Multiplying by c_t^2/m to get rid of the coefficient of the highest derivative yields

$$\theta'' + \frac{\alpha c_t}{m}\theta' + \frac{Cc_t^2}{m}\sin(\theta) = \frac{Ac_t^2}{m}\sin(\omega_0 c_t t).$$

(b) Since we have four different dimensional combinations (although one of those, $\omega_0 c_t$, is not a coefficient) and only one scale (θ is already nondimensional), we have four options:

1) Setting $c_t = m/\alpha$ (which corresponds to the time scale on which damping dissipates the energy) yields

$$\theta'' + \theta' + \frac{Cm}{\alpha^2}\sin(\theta) = \frac{Am}{\alpha^2}\sin\left(\frac{\omega_0 m}{\alpha}t\right).$$

2) Setting $c_t = \sqrt{m/C}$ (which corresponds to the period of undamped oscillations) yields

$$\theta'' + \frac{\alpha}{\sqrt{Cm}}\theta' + \sin(\theta) = \frac{A}{C}\sin\left(\sqrt{\frac{m}{C}}\omega_0 t\right).$$

3) Setting $c_t = \sqrt{m/A}$ (which corresponds to the time scale on which the external forcing pumps the energy into the system) yields

$$\theta'' + \frac{\alpha}{\sqrt{Am}}\theta' + \frac{C}{A}\sin(\theta) = \sin\left(\sqrt{\frac{m}{A}}\omega_0 t\right).$$

4) Setting $c_t = 1/\omega_0$ (which corresponds to the period of external forcing) yields

$$\theta'' + \frac{\alpha}{\omega_0 m}\theta' + \frac{C}{\omega_0^2 m}\sin(\theta) = \frac{A}{\omega_0^2 m}\sin(t).$$

(c) Looking at the four choices above we find the only dimensionless parameter that is analogous to the Rayleigh number is Am/α^2. It corresponds to $c_t = \sqrt{m/A}$.

(d) The quality factor is the property of the oscillator, not the driving, and corresponds to the ratio of the time scale at which damping dissipates energy and the period of oscillation, i.e.,

$$Q = \frac{m/\alpha}{\sqrt{m/C}} = \frac{\sqrt{Cm}}{\alpha}.$$

(e) Rescaling θ is not particularly helpful, because the nonlinear term $\sin(\theta)$ already sets a scale for the angle (2π). Rescaling θ can be used to get rid of one of the coefficients, but will introduce another parameter in the argument of the nonlinear term, so the total number of parameters will not change.

Problem 2

In the new variables the rescaled Swift-Hohenberg equation can be written as

$$\frac{u_0\tau_0}{t_0}\partial_t v = r u_0 v - \xi_0^4\left(q_0^4 + 2\frac{q_0^2}{x_0^2}\partial_y^2 + \frac{1}{x_0^4}\partial_y^4\right)u_0 v - g_0 u_0^3 v^3$$

or, dividing by $u_0\tau_0/t_0$

$$\partial_t v = \frac{r t_0}{\tau_0} v - \frac{\xi_0^4 q_0^4 t_0}{\tau_0}\left(1 + \frac{2}{q_0^2 x_0^2}\partial_y^2 + \frac{1}{q_0^4 x_0^4}\partial_y^4\right)v - \frac{g_0 u_0^2 t_0}{\tau_0}v^3.$$

We can set

$$\frac{r t_0}{\tau_0} = \hat{r}, \qquad \frac{\xi_0^4 q_0^4 t_0}{\tau_0} = 1, \qquad \frac{1}{q_0 x_0} = 1, \qquad \frac{g_0 u_0^2 t_0}{\tau_0} = 1$$

by choosing

$$t_0 = \frac{\tau_0}{\xi_0^4 q_0^4}, \qquad x_0 = \frac{1}{q_0}, \qquad u_0 = \frac{\xi_0^2 q_0^2}{\sqrt{g_0}}, \qquad \hat{r} = \frac{r}{\xi_0^4 q_0^4}$$

which yields the desired result

$$\partial_t v = \hat{r} v - (1 + 2\partial_y^2 + \partial_y^4)v - v^3.$$

Problem 3

(a) We are nondimensionalizing the following equations:

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho_0}\nabla P + \nu\nabla^2\mathbf{v} + \alpha g\theta\hat{z}$$
$$\partial_t\theta + \mathbf{v}\cdot\nabla\theta = \frac{\Delta T}{d}v_z + \kappa\nabla^2\theta, \qquad \nabla\cdot\mathbf{v} = 0$$

The length scale is fixed by the depth d of the fluid layer, hence only four free scales remain: velocity v_0, time t_0, pressure p_0 and temperature θ_0. The number of prefactors is equal to the number of terms (10) minus the number of equations (3), i.e., seven.

By choosing the velocity scale $v_0 = d/t_0$ based on the time and length scales we can ensure that only five prefactors are independent, since terms $\mathbf{v}\cdot\nabla\mathbf{v}$ and $\mathbf{v}\cdot\theta$ will have the same prefactors as $\partial_t\mathbf{v}$ and $\partial_t\theta$, respectively. The number of nondimensional parameters is then equal to the number of independent prefactors (5) minus the number of independent scales (3), i.e.. two.

Alternatively, there are seven parameters: ρ_0, ν, α, g, ΔT, d, and κ, of which two always come in a pair αg. So, effectively, we have only six independent parameters. The number of indepent dimensions is four (length, time, mass, temperature). Hence, according to the Pi theorem, the number of nondimensional parameters is $6 - 4 = 2$.

(b) Replacing $\mathbf{v}\to v_0\mathbf{v}'$, $\mathbf{x}\to d\mathbf{x}'$, $t\to t_0 t'$, $P\to P_0 P'$, $\theta\to\theta_0\theta'$, substituting these into the equations, and dropping the primes yields

$$\frac{v_0}{t_0}\partial_t\mathbf{v} + \frac{v_0^2}{d}\mathbf{v}\cdot\nabla\mathbf{v} = -\frac{P_0}{\rho_0 d}\nabla P + \frac{\nu v_0}{d^2}\nabla^2\mathbf{v} + \alpha g\theta_0\theta\hat{z}$$
$$\frac{\theta_0}{t_0}\partial_t\theta + \frac{v_0\theta_0}{d}\mathbf{v}\cdot\nabla\theta = \frac{\Delta T v_0}{d}v_z + \frac{\kappa\theta_0}{d^2}\nabla^2\theta, \qquad v_0\nabla\cdot\mathbf{v} = 0$$

Dividing these, respectively, by v_0/t_0, θ_0/t_0 and v_0, we find

$$\partial_t\mathbf{v} + \frac{v_0 t_0}{d}\mathbf{v}\cdot\nabla\mathbf{v} = -\frac{P_0 t_0}{\rho_0 v_0 d}\nabla P + \frac{\nu t_0}{d^2}\nabla^2\mathbf{v} + \frac{\alpha g\theta_0 t_0}{v_0}\theta\hat{z}$$
$$\partial_t\theta + \frac{v_0 t_0}{d}\mathbf{v}\cdot\nabla\theta = \frac{\Delta T t_0 v_0}{d\theta_0}v_z + \frac{\kappa t_0}{d^2}\nabla^2\theta, \qquad \nabla\cdot\mathbf{v} = 0$$

(c) We got a total of six nondimensional prefactors

$$\frac{v_0 t_0}{d}, \qquad \frac{P_0 t_0}{\rho_0 v_0 d}, \qquad \frac{\nu t_0}{d^2}, \qquad \frac{\alpha g \theta_0 t_0}{v_0}, \qquad \frac{\Delta T t_0 v_0}{d\theta_0}, \qquad \frac{\kappa t_0}{d^2},$$

or, if we choose $v_0 = d/t_0$, five (since the first of these will become unity), as predicted in part (a).

(d) Setting $t_0 = d^2/\kappa$ reduces the number of prefactors to four. We can choose the scales P_0 and θ_0 to get rid of two more prefactors. In order to get the equations in the same form we wrote them in class, we need to set

$$\frac{P_0 t_0}{\rho_0 v_0 d} = \frac{\nu t_0}{d^2} = \frac{\alpha g \theta_0 t_0}{v_0} = Pr, \qquad \frac{\Delta T t_0 v_0}{d\theta_0} = R,$$

which yields the pressure and temperature scales

$$P_0 = Pr \rho_0 v_0^2 = Pr \rho_0 \left(\frac{\kappa}{d}\right)^2, \qquad \theta_0 = R^{-1} \Delta T.$$

It is trivial to check these have the correct dimensions (since Pr and R are nondimensional).

(e) Solving the four equations above for R, Pr, P_0 and θ_0 we find

$$Pr = \frac{\nu t_0}{d^2} = \frac{\nu}{\kappa}, \qquad P_0 = Pr \rho_0 \left(\frac{\kappa}{d}\right)^2 = \frac{\rho_0 \nu \kappa}{d^2}, \qquad \theta_0 = Pr \frac{v_0}{\alpha g t_0} = \frac{\kappa \nu}{\alpha g d^3}, \qquad R = \frac{\Delta T}{\theta_0} = \frac{\alpha g d^3 \Delta T}{\kappa \nu}$$

in agreement with the expressions we used in class.