Problem 1

(a) The component stripes have wave vectors

$$\mathbf{q}_1 = (1, 0), \qquad \mathbf{q}_2 = \frac{1}{2}(-1, \sqrt{3}), \qquad \mathbf{q}_3 = \frac{1}{2}(-1, -\sqrt{3}).$$

We can remove two of the phases by changing coordinates from $\mathbf{x} = (x, y)$ to $\mathbf{x}' = (x', y')$, such that

$$\mathbf{q}_1 \cdot \mathbf{x} + \phi_1 = \mathbf{q}_1 \cdot \mathbf{x}', \qquad \mathbf{q}_2 \cdot \mathbf{x} + \phi_1 = \mathbf{q}_2 \cdot \mathbf{x}'.$$

Solving these yields

$$x' = x - \frac{\phi_2 q_{1y} - \phi_1 q_{2y}}{q_{1y} q_{2x} - q_{2y} q_{1x}} = x + \phi_1$$

and

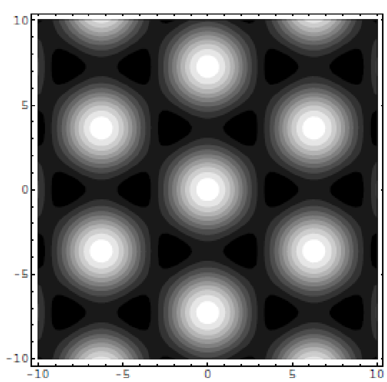
$$y' = y - \frac{\phi_2 q_{1x} - \phi_1 q_{2x}}{q_{1y} q_{2x} - q_{2y} q_{1x}} = y + \frac{\phi_1}{\sqrt{3}} + \frac{2\phi_2}{\sqrt{3}}.$$

In the new coordinates we find

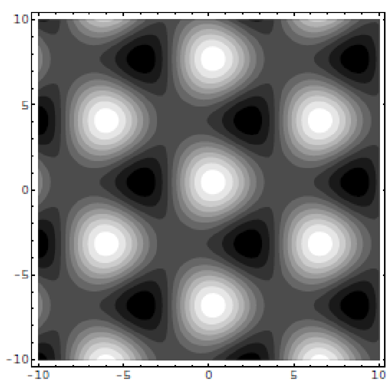
$$u_\phi(\mathbf{x}') = e^{i\mathbf{q}_1 \cdot \mathbf{x}'} + e^{i\mathbf{q}_2 \cdot \mathbf{x}'} + e^{i\mathbf{q}_3 \cdot \mathbf{x}' + i\phi} + c.c.,$$

where $\phi = \phi_1 + \phi_2 + \phi_3$.

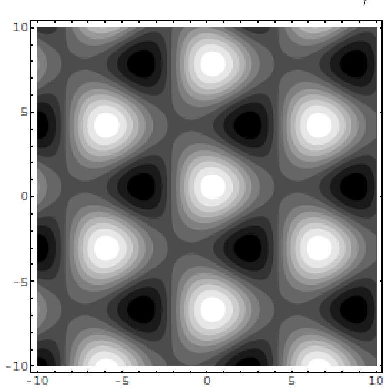
(b) The corresponding patterns for various values of ϕ are:



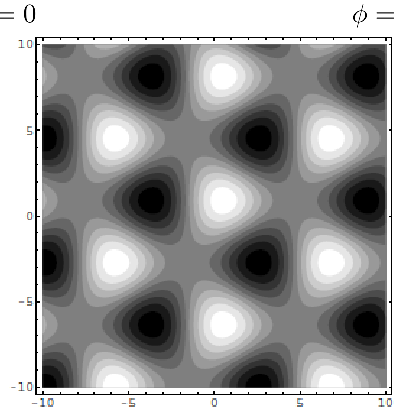
$\phi = 0$



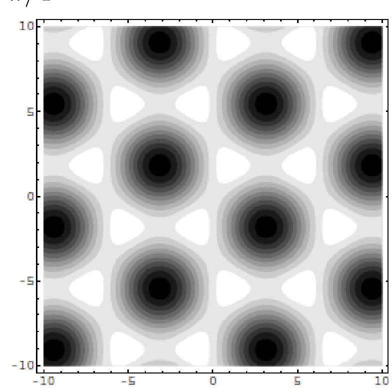
$\phi = \pi/4$



$\phi = \pi/3$



$\phi = \pi/2$



$\phi = \pi$

In particular, we get hexagonal patterns with 6-fold rotational symmetry only for $\phi = 0$ and π.

(c) When $\phi = 0$, the maxima of the pattern (light regions) correspond to the locations where all the three sets of stripes achieve a maximum, while their minima do not coincide. When $\phi = \pi$, the minima of the pattern correspond to the locations where the mimima of the three sets of stripes coincide, while their maxima never do. The patterns are given by

$$u_\phi(\mathbf{x}') = \cos(x') + \cos\left(\frac{-x' + \sqrt{3}y'}{2}\right) \pm \cos\left(\frac{x' + \sqrt{3}y'}{2}\right),$$

where the plus (minus) sign corresponds to $\phi = 0$ ($\phi = \pi$). In each case the periodicity of the pattern in the y direction satisfies $\sqrt{3}\lambda/2 = 2\pi$, so that $\lambda = 4\pi/\sqrt{3}$. The patterns are also periodic with the same period in the directions $\mathbf{e}_1 = (\sqrt{3}/2, 1/2)$ and $\mathbf{e}_2 = (\sqrt{3}/2, -1/2)$ because of the 6-fold rotational symmetry. Hence, the maxima of the pattern (for $\phi = 0$) are located at $\mathbf{x}'_{nm} = n\lambda\mathbf{e}_1 + m\lambda\mathbf{e}_2$ for any integer n and m. The minima of the pattern (for $\phi = \pi$) would coincide with the maxima at $\phi = 0$ after an appropriate shift of the pattern (e.g., by $\mathbf{s} = (\lambda/2)\mathbf{e}_2 = (\pi, -\pi/\sqrt{3})$), i.e., they are at $\mathbf{x}'_{nm} = \mathbf{s} + n\lambda\mathbf{e}_1 + m\lambda\mathbf{e}_2 = n\lambda\mathbf{e}_1 + (m + 1/2)\lambda\mathbf{e}_2$.

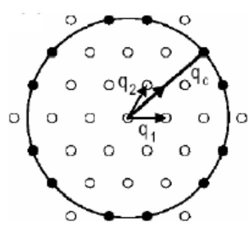
(d) The pattern does not have the inversion symmetry for any ϕ. Instead we have $u_0(\mathbf{x}) = -u_\pi(\mathbf{x} + \mathbf{s})$.

Problem 2

Hexagonal superlattice.

(a) There are twelve wavevectors $\mathbf{q}^n$ making up six component stripes. The angles between odd component stripes (1,3,5) is $\pi/3$, same as the angle between even component stripes (2,4,6). The angle θ between component stripes 1 and 2 (3 and 4 or 5 and 6) will be computed in the next part.

(b) Using the hexagonal grid spanned by basis vectors $\mathbf{q}_1 = (q_s, 0)$ and $\mathbf{q}_2 = (q_s/2, \sqrt{3}q_s/2)$, they can be represented as



$$\begin{aligned}
\mathbf{q}^1 &= 2\mathbf{q}_1 + \mathbf{q}_2, & \mathbf{q}^7 &= -2\mathbf{q}_1 - \mathbf{q}_2,\\
\mathbf{q}^2 &= \mathbf{q}_1 + 2\mathbf{q}_2, & \mathbf{q}^8 &= -\mathbf{q}_1 - 2\mathbf{q}_2,\\
\mathbf{q}^3 &= -\mathbf{q}_1 + 3\mathbf{q}_2, & \mathbf{q}^9 &= \mathbf{q}_1 - 3\mathbf{q}_2,\\
\mathbf{q}^4 &= -2\mathbf{q}_1 + 3\mathbf{q}_2, & \mathbf{q}^{10} &= 2\mathbf{q}_1 - 3\mathbf{q}_2,\\
\mathbf{q}^5 &= -3\mathbf{q}_1 + 2\mathbf{q}_2, & \mathbf{q}^{11} &= 3\mathbf{q}_1 - 2\mathbf{q}_2,\\
\mathbf{q}^6 &= -3\mathbf{q}_1 + \mathbf{q}_2, & \mathbf{q}^{12} &= 3\mathbf{q}_1 - \mathbf{q}_2,
\end{aligned}$$

The angle between $\mathbf{q}^1$ and $\mathbf{q}^2$ is

$$\theta = \arccos\frac{\mathbf{q}^1 \cdot \mathbf{q}^2}{|\mathbf{q}^1||\mathbf{q}^2|} = \arccos\frac{4}{5} = 21.78°.$$

Correspondingly, the angle between $\mathbf{q}^2$ and $\mathbf{q}^3$ is $60° - 21.78° = 38.21°$.

(c) Since $|\mathbf{q}^n| = q_c$ for any n, we can take $n = 1$:

$$q_c = |\mathbf{q}^1| = |2\mathbf{q}_1 + \mathbf{q}_2| = |(5/2, \sqrt{3}/2)|q_s = \sqrt{7}q_s,$$

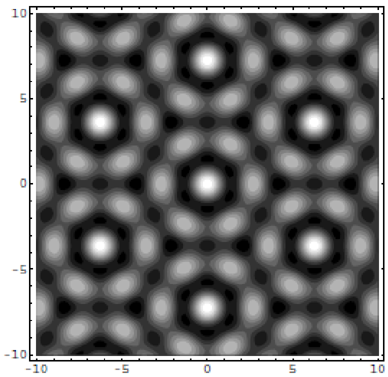
so that

$$\mathbf{q}_1 = \frac{q_c}{\sqrt{7}}(1, 0), \qquad \mathbf{q}_2 = \frac{q_c}{2\sqrt{7}}(1, \sqrt{3}).$$

The pattern (with all six phases set to zero) is given by

$$u(x,y) = \sum_{n=1}^{6} \cos(\mathbf{q}^n \cdot \mathbf{x}) = \cos\left(\frac{q_s}{2}[5x+\sqrt{3}y]\right) + \cos\left(\frac{q_s}{2}[4x+2\sqrt{3}y]\right) + \cos\left(\frac{q_s}{2}[x+3\sqrt{3}y]\right) + \cos\left(\frac{q_s}{2}[5x-\sqrt{3}y]\right) + \cos\left(\frac{q_s}{2}[4x-2\sqrt{3}y]\right) + \cos\left(\frac{q_s}{2}[x-3\sqrt{3}y]\right).$$

and is shown below:



It is easy to see that the pattern is periodic with respect to shifts in the vertical direction as well as along the directions inclined at a $\pm 60^\circ$ angle with respect to the horizontal axis. The period can be found by setting $u(x,y) = u(x, y+\lambda)$. It is easy to see that this requires

$$\frac{\sqrt{3}}{2} q_s \lambda = \frac{\sqrt{3}}{2\sqrt{7}} q_c \lambda = 2\pi$$

such that $\lambda = 4\pi\sqrt{7/3}\, q_c^{-1}$.

Problem 3

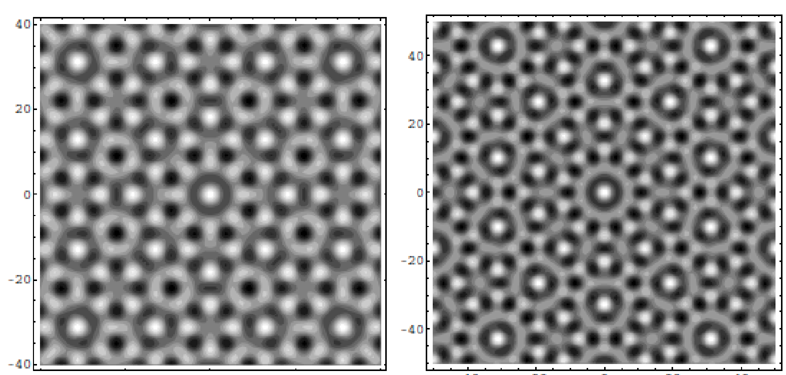
The patterns (with all phases set to zero) can be written as

$$u(\mathbf{x}) = \frac{1}{2}\sum_{n=1}^{N} (e^{i\mathbf{q}^n \cdot \mathbf{x}} + c.c.) = \sum_{n=1}^{N} \cos(\mathbf{q}^n \cdot \mathbf{x}),$$

where $\mathbf{q}^n = (q_c \cos(n\theta), q_c \sin(n\theta))$ and $\theta = \pi/N$. We therefore find

$$u(\mathbf{x}) = \sum_{n=1}^{N} \cos(q_c x \cos(n\theta) + q_c y \sin(n\theta)).$$

The patterns that correspond to $N = 8$ and $N = 10$ are shown below:



$N = 8$ $N = 10$