```
> restart; with(plots): with(linalg): interface(showassumed=0): assume(r>0): assume(mu>0): _EnvExplicit:=true:
```

Problem 1:

Laplacian and the equation for steady state:

> $L := u \rightarrow diff(u, x, x) + diff(u, y, y);$
> $N := u \rightarrow (r-1)*u-2*L(u)-L(L(u))-u\hat{}3;$

$$L := u \rightarrow \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u$$

$$N := u \rightarrow (r-1)\, u - 2\, L(u) - L(L(u)) - u^3 \quad \textbf{(1)}$$

Saturation:

Stripes with arbitrary wavenumber:

> $u1 := a1 \cdot \cos(q \cdot x);$

$$u1 := a1 \cos(q\, x) \quad \textbf{(2)}$$

> $eq := collect(combine(N(u1)), \{\cos(q \cdot x)\});$
> $cf := factor(coeff(eq, \cos(q \cdot x), 1)):$
> $s1 := solve(cf, a1);\ a1 := s1[2]:$

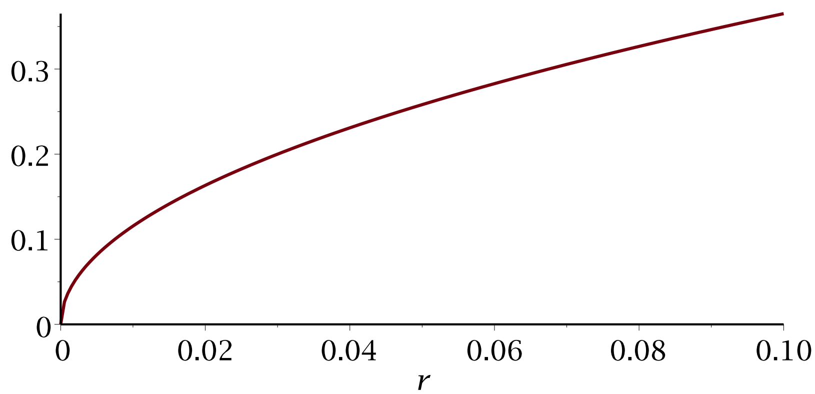
$$eq := \left(-a1 + 2\, a1\, q^2 - a1\, q^4 + a1\, r - \frac{3}{4}\, a1^3\right) \cos(q\, x) - \frac{1}{4}\, a1^3 \cos(3\, q\, x)$$

$$s1 := 0, \frac{2}{3}\sqrt{-3\, q^4 + 6\, q^2 + 3\, r - 3}, -\frac{2}{3}\sqrt{-3\, q^4 + 6\, q^2 + 3\, r - 3} \quad \textbf{(3)}$$

> # The second root can be rewritten as
> $\mathrm{sqrt}\left(\frac{4}{3}\left(r - (1-q^2)^2\right)\right);$

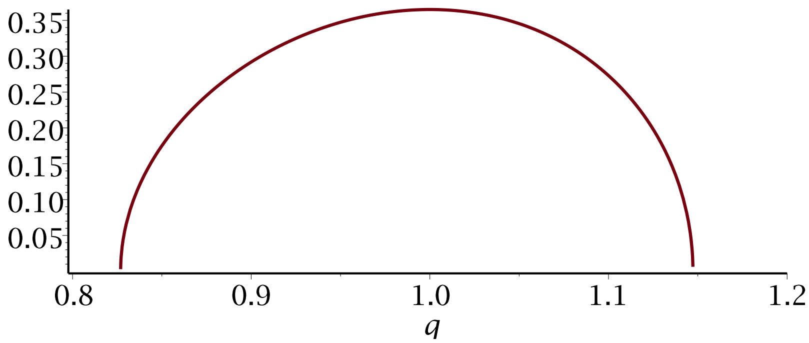
$$\frac{2}{3}\sqrt{-3\, q^4 + 6\, q^2 + 3\, r - 3} \quad \textbf{(4)}$$

> # **Amplitude as function of r:**
> $plot(subs(q = 1, a1), r = 0..0.1)$



> # **Amplitude as function of q:**

$plot(subs(r = 0.1, a1), q = 0.8..1.2);$



Stripes with critical wavenumber:

```
> a1 :='a1':
  u5 := a*cos(x) + b*cos(3*x) + c*cos(5*x);
```

$$u5 := a\cos(x) + b\cos(3\,x) + c\cos(5\,x) \tag{5}$$

Since higher harmonics should correspond to smaller corrections, start expansion from progressively higher powers of sqrt(r):

```
> cfs := {a = a1*μ + a2*μ^2 + a3*μ^3 + a4*μ^4 + a5*μ^5 + a6*μ^6 + a7*μ^7, b = b2*μ^2
      + b3*μ^3 + b4*μ^4 + b5*μ^5 + b6*μ^6 + b7*μ^7, c = c3·μ^3 + c4·μ^4 + c5*μ^5 + c6·μ^6
      + c7*μ^7}:
  eq := collect(combine(subs({r = mu^2} union cfs, N(u5))), {cos(x), cos(3*x),
     cos(5*x), cos(7*x), cos(9*x), cos(11*x), cos(13*x), cos(15*x)}) :
  cf1 := series(factor(coeff(eq, cos(x), 1)), mu, 9) :
  cf3 := series(factor(coeff(eq, cos(3·x), 1)), mu, 9) :
  cf5 := series(factor(coeff(eq, cos(5·x), 1)), mu, 9) :
  # The lowest order terms, i.e. terms of O(μ^2) :
  eq12 := coeff(cf1, mu, 2);
  eq32 := coeff(cf3, mu, 2);
  eq52 := coeff(cf5, mu, 2);
```

$$eq12 := 0$$

$$eq32 := -64\,b2$$

$$eq52 := 0 \tag{6}$$

```
> sl := solve({eq12, eq32, eq52}, {b2});
  cf := sl:
  # Terms of O(μ^3) :
  eq13 := coeff(cf1, mu, 3);
  eq33 := coeff(cf3, mu, 3);
  eq53 := coeff(cf5, mu, 3);
```

$$sl := \{b2 = 0\}$$

$$eq13 := -\frac{3}{4}\,a1^3 + a1$$

$$eq33 := -\frac{1}{4}\,a1^3 - 64\,b3$$

$$eq53 := -576\,c3 \tag{7}$$

```
> sl := solve({eq13, eq33, eq53}, {a1, b3, c3});
  cf := cf union sl[2]:
  # Terms of O(μ^4) :
  eq14 := expand(subs(cf, coeff(cf1, mu, 4)));
  eq34 := expand(subs(cf, coeff(cf3, mu, 4)));
  eq54 := expand(subs(cf, coeff(cf5, mu, 4)));
```

$$sl := \{a1 = 0, b3 = 0, c3 = 0\}, \left\{a1 = \frac{2}{3}\sqrt{3}, b3 = -\frac{1}{288}\sqrt{3}, c3 = 0\right\}, \left\{a1 = -\frac{2}{3}\sqrt{3}, b3 = \frac{1}{288}\sqrt{3}, c3 = 0\right\}$$

$$eq14 := -2\,a2$$

$$eq34 := -a2 - 64\,b4$$

$$eq54 := -576\,c4 \tag{8}$$

```
> sl := solve({eq14, eq34, eq54}, {a2, b4, c4});
  cf := cf union sl;
  # Terms of O(μ^5) :
  eq15 := expand(subs(cf, coeff(cf1, mu, 5)));
  eq35 := expand(subs(cf, coeff(cf3, mu, 5)));
  eq55 := expand(subs(cf, coeff(cf5, mu, 5)));
```

$$sl := \{a2 = 0, b4 = 0, c4 = 0\}$$

$$cf := \left\{a1 = \frac{2}{3}\sqrt{3}, a2 = 0, b2 = 0, b3 = -\frac{1}{288}\sqrt{3}, b4 = 0, c3 = 0, c4 = 0\right\}$$

$$eq15 := -2\,a3 + \frac{1}{288}\sqrt{3}$$

$$eq35 := -a3 + \frac{1}{288}\sqrt{3} - 64\,b5$$

$$eq55 := \frac{1}{288}\sqrt{3} - 576\,c5 \tag{9}$$

```
> sl := solve({eq15, eq35, eq55}, {a3, b5, c5});
  cf := cf union sl;
  # Terms of O(μ^6) :
  eq16 := expand(subs(cf, coeff(cf1, mu, 6)));
  eq36 := expand(subs(cf, coeff(cf3, mu, 6)));
  eq56 := expand(subs(cf, coeff(cf5, mu, 6)));
```

$$sl := \left\{a3 = \frac{1}{576}\sqrt{3}, b5 = \frac{1}{36864}\sqrt{3}, c5 = \frac{1}{165888}\sqrt{3}\right\}$$

$$cf := \left\{a1 = \frac{2}{3}\sqrt{3},\ a2 = 0,\ a3 = \frac{1}{576}\sqrt{3},\ b2 = 0,\ b3 = -\frac{1}{288}\sqrt{3},\ b4 = 0,\ b5 = \frac{1}{36864}\sqrt{3},\ c3 = 0,\ c4 = 0,\ c5 = \frac{1}{165888}\sqrt{3}\right\}$$

$$eq16 := -2\ a4$$

$$eq36 := -a4 - 64\ b6$$

$$eq56 := -576\ c6 \tag{10}$$

> *sl* := *solve*({*eq16*, *eq36*, *eq56*}, {a4, b6, c6});
> *cf* := cf **union** *sl*;
> \# **Terms of O(μ⁷) :**
> *eq17* := *expand*(*subs*(*cf*, *coeff*(*cf1*, *mu*, *7*)));
> *eq37* := *expand*(*subs*(*cf*, *coeff*(*cf3*, mu, 7)));
> *eq57* := *expand*(*subs*(*cf*, *coeff*(*cf5*, *mu*, *7*)));

$$sl := \{a4 = 0,\ b6 = 0,\ c6 = 0\}$$

$$cf := \left\{a1 = \frac{2}{3}\sqrt{3},\ a2 = 0,\ a3 = \frac{1}{576}\sqrt{3},\ a4 = 0,\ b2 = 0,\ b3 = -\frac{1}{288}\sqrt{3},\ b4 = 0,\ b5 = \frac{1}{36864}\sqrt{3},\ b6 = 0,\ c3 = 0,\ c4 = 0,\ c5 = \frac{1}{165888}\sqrt{3},\ c6 = 0\right\}$$

$$eq17 := -2\ a5 - \frac{13}{221184}\sqrt{3}$$

$$eq37 := -a5 - \frac{1}{663552}\sqrt{3} - 64\ b7$$

$$eq57 := -\frac{11}{331776}\sqrt{3} - 576\ c7 \tag{11}$$

> *sl* := *solve*({*eq17*, *eq37*, *eq57*}, {a5, b7, c7});
> *cf* := cf **union** sl;

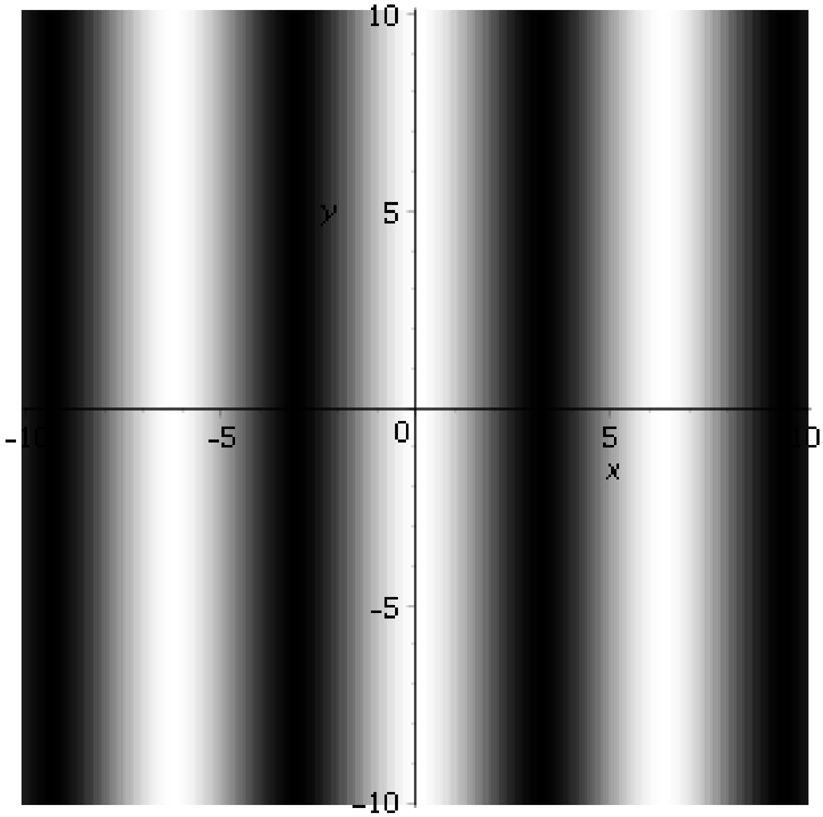
$$sl := \left\{a5 = -\frac{13}{442368}\sqrt{3},\ b7 = \frac{37}{84934656}\sqrt{3},\ c7 = -\frac{11}{191102976}\sqrt{3}\right\}$$

$$cf := \left\{a1 = \frac{2}{3}\sqrt{3},\ a2 = 0,\ a3 = \frac{1}{576}\sqrt{3},\ a4 = 0,\ a5 = -\frac{13}{442368}\sqrt{3},\ b2 = 0,\ b3 = -\frac{1}{288}\sqrt{3},\ b4 = 0,\ b5 = \frac{1}{36864}\sqrt{3},\ b6 = 0,\ b7 = \frac{37}{84934656}\sqrt{3},\ c3 = 0,\ c4 = 0,\ c5 = \frac{1}{165888}\sqrt{3},\ c6 = 0,\ c7 = -\frac{11}{191102976}\sqrt{3}\right\} \tag{12}$$

> \# **After throwing away terms of O(μ⁶) and higher, we obtain :**
> *us* := *subs*(μ = sqrt(*r*), *subs*(*cf*, *subs*({*a6* = 0, *a7* = 0, *b6* = 0, *b7* = 0, *c6* = 0, *c7* = 0}, *subs*(*cfs*, *u5*))));
> *densityplot*(*subs*(r = 0.1, *us*), *x* = -10..10, *y* = -10..10, *grid* = [100, 100], *style* = *PATCHNOGRID*);

$$us := \left(-\frac{13}{442368}\sqrt{3}\ r^{5/2} + \frac{1}{576}\sqrt{3}\ r^{3/2} + \frac{2}{3}\sqrt{3}\sqrt{r}\right)\cos(x)$$

$$+\left(\frac{1}{36864}\sqrt{3}\,r^{5/2}-\frac{1}{288}\sqrt{3}\,r^{3/2}\right)\cos(3\,x)+\frac{1}{165888}\sqrt{3}\,r^{5/2}\cos(5\,x)$$



> $evalf(us);$

$$\left(-0.00005090029229\,r^{5/2}+0.003007032653\,r^{3/2}+1.154700539\sqrt{r}\right)\cos(x)+\left(0.00004698488520\,r^{5/2}-0.006014065305\,r^{3/2}\right)\cos(3.\,x)+0.00001044108560\,r^{5/2}\cos(5.\,x) \quad \textbf{(13)}$$

Problem 2:

Equation for steady state:

> $N := u \rightarrow (r-1)*u-2*L(u)-L(L(u))-g\cdot u^2-u\text{^}3;$

$$N := u \rightarrow (r-1)\,u-2\,L(u)-L(L(u))-g\,u^2-u^3 \quad \textbf{(14)}$$

> $u3 := a1*\cos(x)+a2*\cos(2*x)+a3*\cos(3*x);$

$$u3 := a1\cos(x)+a2\cos(2\,x)+a3\cos(3\,x) \quad \textbf{(15)}$$

Since higher harmonics should correspond to smaller corrections, start expansion from progressively higher powers of sqrt(r):

> $cfs := \{a1 = a11*\mu+a12*\mu^2+a13*\mu^3,\ a2 = a22*\mu^2+a23*\mu^3,\ a3 = a33\cdot\mu^3\}:$

```
eq := collect(combine(subs({r = mu^2} union cfs, N(u3))), {cos(x), cos(3*x),
    cos(5*x), cos(7*x), cos(9*x), cos(11*x), cos(13*x), cos(15*x)}) :
cf1 := series(factor(coeff(eq, cos(x), 1)), mu, 4);
cf2 := series(factor(coeff(eq, cos(2·x), 1)), mu, 3);
cf3 := series(factor(coeff(eq, cos(3·x), 1)), mu, 4);
cf4 := series(factor(coeff(eq, cos(4·x), 1)), mu, 5);
```

$$cf1 := \left(-\frac{3}{4}\,a11^3 - a11\,a22\,g + a11\right)\mu^3 + O(\mu^4)$$

$$cf2 := \left(-\frac{1}{2}\,a11^2\,g - 9\,a22\right)\mu^2 + O(\mu^3)$$

$$cf3 := \left(-\frac{1}{4}\,a11^3 - a11\,a22\,g - 64\,a33\right)\mu^3 + O(\mu^4)$$

$$cf4 := \left(-\frac{3}{4}\,a11^2\,a22 - a11\,a33\,g - \frac{1}{2}\,a22^2\,g\right)\mu^4 + O(\mu^5) \qquad \textbf{(16)}$$

```
> # Use only the lowest order terms in mu, i.e.
eq13 := coeff(cf1, mu, 3);
eq22 := coeff(cf2, mu, 2);
eq33 := coeff(cf3, mu, 3);
```

$$eq13 := -\frac{3}{4}\,a11^3 - a11\,a22\,g + a11$$

$$eq22 := -\frac{1}{2}\,a11^2\,g - 9\,a22$$

$$eq33 := -\frac{1}{4}\,a11^3 - a11\,a22\,g - 64\,a33 \qquad \textbf{(17)}$$

```
> sl := solve({eq13, eq22, eq33}, {a11, a22, a33});
```

$$sl := \{a11 = 0,\ a22 = 0,\ a33 = 0\},\ \left\{a11 = -\frac{6}{\sqrt{-2\,g^2+27}},\ a22 = \frac{2\,g}{2\,g^2-27},\ a33 = \frac{3}{32}\,\frac{2\,g^2-9}{(2\,g^2-27)\sqrt{-2\,g^2+27}}\right\},\ \left\{a11 = \frac{6}{\sqrt{-2\,g^2+27}},\ a22 = \frac{2\,g}{2\,g^2-27},\ a33 = -\frac{3}{32}\,\frac{2\,g^2-9}{(2\,g^2-27)\sqrt{-2\,g^2+27}}\right\} \qquad \textbf{(18)}$$

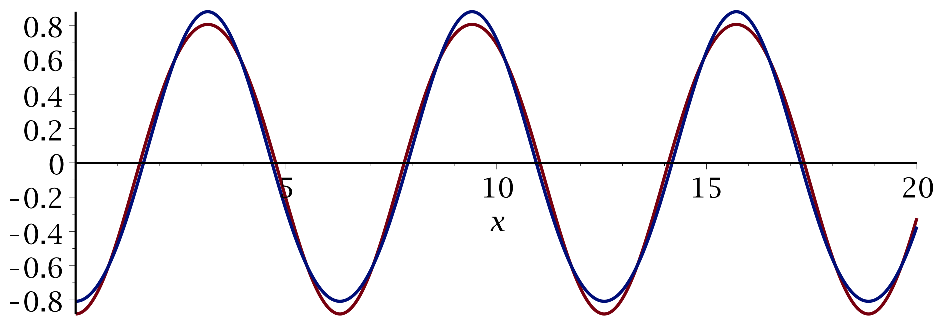
```
> # First nontrivial solution (to leading orders in g):
u1 := collect(convert(series(subs({mu = sqrt(r), a12 = 0, a13 = 0, a23 = 0},
    subs(sl[2], subs(cfs, u3))), g, 3), polynom), {cos(x), cos(2·x), cos(3·x)});
```

$$u1 := \left(-\frac{2}{3}\sqrt{3}\sqrt{r} - \frac{2}{81}\sqrt{3}\sqrt{r}\,g^2\right)\cos(x) - \frac{2}{27}\,r\cos(2\,x)\,g + \left(\frac{1}{288}\sqrt{3}\,r^{3/2} - \frac{1}{2592}\sqrt{3}\,r^{3/2}\,g^2\right)\cos(3\,x) \qquad \textbf{(19)}$$

```
> # Second nontrivial solution (to leading orders in g):
u2 := collect(convert(series(subs({mu = sqrt(r), a12 = 0, a13 = 0, a23 = 0},
    subs(sl[3], subs(cfs, u3))), g, 3), polynom), {cos(x), cos(2·x), cos(3·x)})
```

$$u2 := \left(\frac{2}{3}\sqrt{3}\sqrt{r} + \frac{2}{81}\sqrt{3}\sqrt{r}\,g^2\right)\cos(x) - \frac{2}{27}\,r\cos(2\,x)\,g + \left(-\frac{1}{288}\sqrt{3}\,r^{3/2} + \frac{1}{2592}\sqrt{3}\,r^{3/2}\,g^2\right)\cos(3\,x) \quad (20)$$

> *# The two solutions are not negatives of each other since the coefficients of the cos(2x) terms are equal*
plot(subs({r = 0.5, g = 1}, [*u1*, -*u2*]), *x* = 0 ..20, *numpoints* = 400);



Problem 3:

>

> *qc:=Pi/sqrt(2);*

$$qc := \frac{1}{2}\,\pi\sqrt{2} \quad (21)$$

> **# x- and z-components of the Navier-Stokes equation:**

```
> nsex:=tau*diff(ux(x,z,t),t)+ux(x,z,t)*diff(ux(x,z,t),x)+uz(x,z,t)*diff(ux(x,z,t),z)-Pr*(-diff(p(x,z,t),x)+diff(ux(x,z,t),x,x)+diff(ux(x,z,t),z,z));
nsez:=tau*diff(uz(x,z,t),t)+ux(x,z,t)*diff(uz(x,z,t),x)+uz(x,z,t)*diff(uz(x,z,t),z)-Pr*(-diff(p(x,z,t),z)+diff(uz(x,z,t),x,x)+diff(uz(x,z,t),z,z)+theta(x,z,t));
```

$$nsex := \tau\left(\frac{\partial}{\partial t}\,ux(x,z,t)\right) + ux(x,z,t)\left(\frac{\partial}{\partial x}\,ux(x,z,t)\right) + uz(x,z,t)\left(\frac{\partial}{\partial z}\,ux(x,z,t)\right) - Pr\left(-\left(\frac{\partial}{\partial x}\,p(x,z,t)\right) + \frac{\partial^2}{\partial x^2}\,ux(x,z,t) + \frac{\partial^2}{\partial z^2}\,ux(x,z,t)\right)$$

$$nsez := \tau\left(\frac{\partial}{\partial t}\,uz(x,z,t)\right) + ux(x,z,t)\left(\frac{\partial}{\partial x}\,uz(x,z,t)\right) + uz(x,z,t)\left(\frac{\partial}{\partial z}\,uz(x,z,t)\right) - Pr\left(-\left(\frac{\partial}{\partial z}\,p(x,z,t)\right) + \frac{\partial^2}{\partial x^2}\,uz(x,z,t) + \frac{\partial^2}{\partial z^2}\,uz(x,z,t) + \theta(x,z,t)\right) \quad (22)$$

> **# Heat advection-diffusion equation and the y-component of the vorticity equation:**

```
> hade:=tau*diff(theta(x,z,t),t)+ux(x,z,t)*diff(theta(x,z,t),x)+uz(x,z,t)*diff(theta(x,z,t),z)-(R*uz(x,z,t)+diff(theta(x,z,t),x,x)+diff(theta(x,z,t),z,z));
omega:=expand(diff(nsex,z)-diff(nsez,x));
```

$$hade := \tau\left(\frac{\partial}{\partial t}\,\theta(x,z,t)\right) + ux(x,z,t)\left(\frac{\partial}{\partial x}\,\theta(x,z,t)\right) + uz(x,z,t)\left(\frac{\partial}{\partial z}\,\theta(x,z,t)\right)$$

$$- R\, uz(x, z, t) - \left(\frac{\partial^2}{\partial x^2} \theta(x, z, t)\right) - \left(\frac{\partial^2}{\partial z^2} \theta(x, z, t)\right)$$

$$\omega := \tau \left(\frac{\partial^2}{\partial z\, \partial t} ux(x, z, t)\right) + \left(\frac{\partial}{\partial z} ux(x, z, t)\right)\left(\frac{\partial}{\partial x} ux(x, z, t)\right) + ux(x, z, t)\left(\frac{\partial^2}{\partial z\, \partial x} ux(x, z, t)\right) + \left(\frac{\partial}{\partial z} uz(x, z, t)\right)\left(\frac{\partial}{\partial z} ux(x, z, t)\right) + uz(x, z, t)\left(\frac{\partial^2}{\partial z^2} ux(x, z, t)\right) - Pr\left(\frac{\partial^3}{\partial z\, \partial x^2} ux(x, z, t)\right) - Pr\left(\frac{\partial^3}{\partial z^3} ux(x, z, t)\right) - \tau\left(\frac{\partial^2}{\partial x\, \partial t} uz(x, z, t)\right) - \left(\frac{\partial}{\partial x} ux(x, z, t)\right)\left(\frac{\partial}{\partial x} uz(x, z, t)\right) - ux(x, z, t)\left(\frac{\partial^2}{\partial x^2} uz(x, z, t)\right) - \left(\frac{\partial}{\partial x} uz(x, z, t)\right)\left(\frac{\partial}{\partial z} uz(x, z, t)\right) - uz(x, z, t)\left(\frac{\partial^2}{\partial z\, \partial x} uz(x, z, t)\right) + Pr\left(\frac{\partial^3}{\partial x^3} uz(x, z, t)\right) + Pr\left(\frac{\partial^3}{\partial z^2\, \partial x} uz(x, z, t)\right) + Pr\left(\frac{\partial}{\partial x} \theta(x, z, t)\right) \quad \textbf{(23)}$$

```
> # Ansatz: Linear equation plus the leading order correction (same as in d'Arcy convection, with scales alpha, beta, delta):
> psi(x,z,t):=alpha*X(t)*cos(Pi*z)*sin(qc*x);
  ansatz:={ux(x,z,t)=-diff(psi(x,z,t),z),uz(x,z,t)=diff(psi(x,z,t),x),theta(x,z,t)=beta*Y(t)*cos(Pi*z)*cos(qc*x)+delta*Z(t)*sin(2*Pi*z)};
  eq1:=simplify(eval(subs(ansatz,omega)));
  eq2:=simplify(eval(subs(ansatz,hade)));
```

$$\psi(x, z, t) := \alpha\, X(t) \cos(\pi z) \sin\left(\frac{1}{2}\pi\sqrt{2}\, x\right)$$

$$ansatz := \left\{\theta(x, z, t) = \beta\, Y(t)\cos(\pi z)\cos\left(\frac{1}{2}\pi\sqrt{2}\, x\right) + \delta\, Z(t)\sin(2\pi z),\ ux(x, z, t) = \alpha\, X(t)\sin(\pi z)\,\pi\sin\left(\frac{1}{2}\pi\sqrt{2}\, x\right),\ uz(x, z, t) = \frac{1}{2}\alpha\, X(t)\cos(\pi z)\cos\left(\frac{1}{2}\pi\sqrt{2}\, x\right)\pi\sqrt{2}\right\}$$

$$eq1 := \frac{1}{4}\cos(\pi z)\,\pi\sin\left(\frac{1}{2}\pi\sqrt{2}\, x\right)\left(9\, Pr\,\alpha\, X(t)\,\pi^3 + 6\,\tau\,\alpha\left(\frac{d}{dt} X(t)\right)\pi - 2\, Pr\,\beta\, Y(t)\sqrt{2}\right)$$

$$eq2 := \left(\frac{d}{dt} Y(t)\right)\cos(\pi z)\cos\left(\frac{1}{2}\pi\sqrt{2}\, x\right)\beta\,\tau + \left(\frac{d}{dt} Z(t)\right)\sin(2\pi z)\,\delta\,\tau - \frac{1}{2}\alpha\, X(t)\sin(\pi z)\,\pi^2\,\beta\, Y(t)\cos(\pi z)\sqrt{2} + \cos(\pi z)\cos\left(\frac{1}{2}\pi\sqrt{2}\, x\right) X(t)\,\pi^2\sqrt{2}\, Z(t)\cos(2\pi z)\,\alpha\,\delta \quad \textbf{(24)}$$

$$-\frac{1}{2}\,R\,\alpha\,X(t)\cos(\pi z)\cos\left(\frac{1}{2}\,\pi\sqrt{2}\,x\right)\pi\sqrt{2}$$
$$+\frac{3}{2}\,\beta\,Y(t)\cos(\pi z)\cos\left(\frac{1}{2}\,\pi\sqrt{2}\,x\right)\pi^2+4\,\delta\,Z(t)\sin(2\,\pi z)\,\pi^2$$

> **# Fourier coefficients:**

```
> eqx:=collect(expand(int(int(-eq1*cos(Pi*z)*sin(qc*x),z=0..2),x=0..2*Pi/qc)*2/sqrt(2)
/3/Pi^2/alpha/tau),{X(t),Y(t)});
eqy:=expand(int(int(-eq2*cos(Pi*z)*cos(qc*x),z=0..2),x=0..2*Pi/qc)/beta/sqrt(2)
/tau);
eqz:=expand(int(int(-eq2*sin(2*Pi*z),z=0..2),x=0..2*Pi/qc)/2/sqrt(2)/tau/delta);
```

$$eqx := -\frac{3}{2}\,\frac{\pi^2\,Pr\,X(t)}{\tau}+\frac{1}{3}\,\frac{\sqrt{2}\,Pr\,\beta\,Y(t)}{\pi\,\alpha\,\tau}-\left(\frac{d}{dt}\,X(t)\right)$$

$$eqy := -\frac{3}{2}\,\frac{Y(t)\,\pi^2}{\tau}+\frac{1}{2}\,\frac{\sqrt{2}\,X(t)\,\pi\,R\,\alpha}{\beta\,\tau}-\left(\frac{d}{dt}\,Y(t)\right)-\frac{1}{2}\,\frac{\sqrt{2}\,X(t)\,\pi^2\,Z(t)\,\alpha\,\delta}{\beta\,\tau}$$

$$eqz := -\left(\frac{d}{dt}\,Z(t)\right)+\frac{1}{4}\,\frac{\sqrt{2}\,\alpha\,X(t)\,\pi^2\,\beta\,Y(t)}{\tau\,\delta}-\frac{4\,\pi^2\,Z(t)}{\tau} \qquad \textbf{(25)}$$

```
> scales:=solve({(4/27)*R/Pi^4=r,(3/2)*Pi^2/tau=1,(2/9)*Pr*sqrt(2)*beta/(Pi^3*
alpha)=Pr,(1/2)*sqrt(2)*Pi^2*alpha*delta/(beta*tau)=1,(1/4)*sqrt(2)*Pi^2*alpha*
beta/(tau*delta)=4*Pi^2/tau},{beta,tau,delta,alpha,R});
```

$$scales := \left\{R=\frac{27}{4}\,\pi^4\,r,\ \alpha=2\sqrt{6},\ \beta=\frac{9}{2}\,\pi^3\sqrt{6}\sqrt{2},\ \delta=\frac{27}{4}\,\pi^3,\ \tau=\frac{3}{2}\,\pi^2\right\},\ \left\{R=\frac{27}{4}\,\pi^4\,r,\ \alpha=-2\sqrt{6},\ \beta=-\frac{9}{2}\,\pi^3\sqrt{6}\sqrt{2},\ \delta=\frac{27}{4}\,\pi^3,\ \tau=\frac{3}{2}\,\pi^2\right\} \qquad \textbf{(26)}$$

> **# The famous Lorenz equations:**

```
expand(simplify(subs(scales,eqx)));
expand(simplify(subs(scales,eqy)));
collect(expand(simplify(subs(scales,eqz))),{Y(t),X(t)});
```

$$-Pr\,X(t)+Pr\,Y(t)-\left(\frac{d}{dt}\,X(t)\right)$$

$$-Y(t)+X(t)\,r-\left(\frac{d}{dt}\,Y(t)\right)-X(t)\,Z(t)$$

$$-\left(\frac{d}{dt}\,Z(t)\right)+\frac{8}{3}\,X(t)\,Y(t)-\frac{8}{3}\,Z(t) \qquad \textbf{(27)}$$

> **# Nonlinearly saturated solution:**

```
subs(scales,ansatz);
```

$$\left\{\theta(x,z,t)=\frac{9}{2}\,\pi^3\sqrt{6}\sqrt{2}\,Y(t)\cos(\pi z)\cos\left(\frac{1}{2}\,\pi\sqrt{2}\,x\right)+\frac{27}{4}\,\pi^3\,Z(t)\sin(2\,\pi z), \qquad \textbf{(28)}\right.$$

$$ux(x,z,t)=2\sqrt{6}\,X(t)\sin(\pi z)\,\pi\sin\left(\frac{1}{2}\,\pi\sqrt{2}\,x\right),\ uz(x,z,t)$$

$$= \sqrt{6}\, X(t) \cos(\pi z) \cos\left(\frac{1}{2}\pi\sqrt{2}\, x\right) \pi\sqrt{2}\}$$