Homework #9 Solutions

Note Title 4/20/2006

Problem 1:

(a) cross-roll instability does not describe a long-wavelength modulation of the stripe pattern and so cannot be described within the framework of a single-stripe amplitude equation.

(b) $$u = A_0 e^{iqx} = A_0 e^{i(q-q_c)x} e^{iqx} \equiv \underbrace{A_0 e^{ikx}}_{\text{Amplitude } A(x)} e^{iq_c x}$$

Substituting into amplitude equation:

$$0 = \varepsilon A_0 + \xi_0^2 \partial_x^2 A_0 - g_0 |A|^2 A_0 = \left(\varepsilon + \xi_0^2 k^2 - g_0 |A_0|^2\right) A_0$$

$$\Rightarrow A_0 = \pm \sqrt{\frac{\varepsilon - \xi_0^2 k^2}{g_0}} = \pm \sqrt{\frac{\varepsilon}{g_0}} \sqrt{1 - K^2}, \quad K \equiv \frac{\xi_0 (q - q_c)}{\sqrt{\varepsilon}} = \frac{\xi_0 k}{\sqrt{\varepsilon}}$$

(c) Linearizing $\partial_T \bar{A} = \bar{A} + \left(\partial_X - \frac{i}{2}\partial_Y^2\right)^2 \bar{A} - |\bar{A}|^2 A$ about $\bar{A} = \bar{A}_K$ we mostly need to worry about the nonlinear term $|\bar{A}|^2 \bar{A}$:

$$|\bar{A}_k+\delta\bar{A}|^2(\bar{A}_k+\delta\bar{A}) = (\bar{A}_k^*+\delta\bar{A}^*)(\bar{A}_k+\delta A)(\bar{A}_k+\delta A) = A_k^* A_k A_k +$$

$$+ \bar{A}_k^*\bar{A}_k\delta\bar{A} + \bar{A}_k^*\bar{A}_k\delta\bar{A} + A_k^2\delta\bar{A}^* + o(\delta\bar{A}^2), \text{ so}$$

$$\Rightarrow \partial_T\delta\bar{A} = \delta\bar{A} + (\partial_x - \tfrac{i}{2}\partial_y^2)^2\delta\bar{A} - 2|A_k|\delta\bar{A} - A_k^2\delta\bar{A}^*$$

(d) Take $\delta\bar{A} = e^{ikx}[\delta a_+ e^{i\vec{Q}\vec{x}} + \delta a_-^* e^{-i\vec{Q}\vec{x}}]$

① $$(\partial_x - \tfrac{i}{2}\partial_y^2)^2\delta\bar{A} = \delta a_+(i(k+Q_x) - \tfrac{i}{2}(iQ_y)^2)^2 e^{ikx}e^{i\vec{Q}\cdot\vec{x}} +$$

$$+\delta a_-^*(i(k-Q_x) - \tfrac{i}{2}(-iQ_y)^2)^2 e^{ikx}e^{-i\vec{Q}\cdot\vec{x}}$$

$$= -\delta a_+(k+Q_x+\tfrac{Q_y^2}{2})^2 e^{ikx}e^{i\vec{Q}\vec{x}} - \delta a_-^*(k-Q_x+\tfrac{Q_y^2}{2})^2 e^{ikx}e^{-i\vec{Q}\cdot\vec{x}}$$

② $$-2|\bar{A}_k|^2\delta\bar{A} = -2(1-k^2)e^{ikx}[\delta a_+ e^{i\vec{Q}\cdot\vec{x}} + \delta a_-^* e^{-i\vec{Q}\vec{x}}]$$

③ $$\bar{A}_k^2\delta\bar{A}^* = -(1-k^2)e^{2ikx}e^{-ikx}[\delta a_+^* e^{-i\vec{Q}\cdot\vec{x}} + \delta a_- e^{i\vec{Q}\vec{x}}]$$

Collecting terms with $e^{i\vec{Q}\vec{x}}$ we get

$$\delta\dot{a}_+ = -(1-k^2)\delta a_+ - (Q_x^2+\tfrac{Q_y^4}{4}+2kQ_x+kQ_y^2+Q_xQ_y^2)\delta a_+ - (1-k^2)\delta a_-$$

$$\equiv -(P^2+U_+)\delta a_+ - P^2\delta a_-$$

and for $e^{-i\tilde{Q}\tilde{x}}$:

$$\delta\dot{a}_- = -(1-K^2)\delta a_- - \left(Q_x^2 + \frac{Q_y^4}{4} - 2KQ_x + KQ_y^2 - Q_xQ_y^2\right)\delta a_- - (1-K^2)\delta a_+$$

$$\equiv -P^2\delta a_+ - (P^2+U_-)\delta a_-,$$

where $P^2 = 1-K^2$ and $U_\pm = Q_x^2 + \frac{Q_y^4}{4} \pm 2KQ_x + KQ_y^2 \pm Q_xQ_y^2$

(e) Finding the eigenvalues of the Jacobian we get

$$\sigma_K^\pm(\vec{Q}) = -(1-K^2) - Q_x^2 - KQ_y^2 - \frac{Q_y^4}{4} \pm \left[1+K^4-2K^2+4K^2Q_x^2+4KQ_x^2+4KQ_x^2Q_y^2+Q_x^2Q_y^4\right]^{\frac{1}{2}}$$

where $\sigma_K^+ > \sigma_K^-$ and therefore determines the growth rate of long-wavelength disturbances.

(f) For $Q_x=0$, $\sigma_K(\vec{Q}) = -\frac{1}{4}Q_y^2(4K+Q_y^2)$

$\sigma_K(0) = 0$, so this is a type-II_s instability.

For $K<0$ the maximal growth rate $\sigma_{K\max} = K^2$ occurs at $Q_y = \sqrt{-2K}$

In terms of q: $\kappa<0 \Leftrightarrow q<q_c$ leads to instability.

(g) For $Q_y=0$, $\sigma_\kappa(\vec{Q}) = -(1-\kappa^2)-Q_x^2+\sqrt{(1-\kappa^2)^2-(2\kappa Q_x)^2}$

$$\approx -\frac{1-3\kappa^2}{1-\kappa^2}Q_x^2 - \frac{2\kappa^4}{(1-\kappa^2)^3}Q_x^4+\cdots,$$

So we again find a type-II_s instability ($\sigma_\kappa(0)=0$).

The largest growth rate $\sigma_{\kappa,max} = \frac{(3\kappa^2-1)^2}{4\kappa^2}$ occurs at

$$Q_x^2 = 3\frac{(\kappa^2+1)(3\kappa^2-1)}{4\kappa^2}.$$

In terms of the original units $\sigma_\kappa<0 \Leftrightarrow 1-3\kappa^2>0 \Leftrightarrow |\kappa|>\frac{1}{3}$

which means $|q-q_c|>\frac{\varepsilon^{1/2}}{\sqrt{3}\,\xi_0}$ or $(q-q_c)^2>\frac{\varepsilon}{3\xi_0^2}$

Problem 2

(a) For $\partial_t u = ru - (1+\nabla^2)^2 u + \vec{\nabla}\cdot((\vec{\nabla}u)^2 \vec{\nabla}u)$ we found previously for

Single stripes: $u_1(x) = \frac{2}{\sqrt{3}}\sqrt{r}\cos x = \frac{\sqrt{r}}{\sqrt{3}}(e^{ix} + c.c.)$

For double stripes substitution of the ansatz

$$u_2(x,y) = a\big(\cos x + \cos(\underbrace{x\cos\theta + y\sin\theta}_{z})\big)$$

yields

$$u_2(x,y) = \frac{2}{\sqrt{4\cos^2\theta + 5}}\sqrt{r}\,(\cos x + \cos z) =$$

$$= (4\cos^2\theta + 5)^{-1/2}\sqrt{r}\,(e^{ix} + e^{iz} + c.c.)$$

From the corresponding amplitude equations we would get

$$u_1 = \frac{\sqrt{r}}{\sqrt{g_0}}(e^{ix} + c.c.) \text{ and } u_2 = (1+G(\theta))^{-1/2}\frac{\sqrt{r}}{\sqrt{g_0}},$$

Equating the two expressions for u_1 we again find $g_0 = 3$, so

$$1 + G(\theta) = \frac{4}{3}\cos^2\theta + \frac{5}{3} \Rightarrow G(\theta) = \frac{4}{3}\cos^2\theta + \frac{2}{3}$$

(b) Since $G(\frac{\pi}{2}) = \frac{2}{3} < 1$, the stripe state is unstable while the square lattice state is stable. This agrees with the previous calculation based on Galerkin expansion.

Problem 3

Linearizing

$$\begin{cases} \partial_t A_1 = A_1 - \left(|A_1|^2 + G(\frac{\pi}{3})|A_2|^2 + G(\frac{\pi}{3})|A_3|^2\right)A_1 \\ \partial_t A_2 = A_2 - \left(|A_2|^2 + G(\frac{\pi}{3})|A_1|^2 + G(\frac{\pi}{3})|A_3|^2\right)A_2 \\ \partial_t A_3 = A_3 - \left(|A_3|^2 + G(\frac{\pi}{3})|A_1|^2 + G(\frac{\pi}{3})|A_2|^2\right)A_3 \end{cases}$$

about the hex. state $A_1 = A_2 = A_3 = A_H = (1 + 2G(\frac{\pi}{3}))^{-1/2}$ and assuming δA_i to be real (we are not interested in shifts of the stripes) we get

$$\frac{d}{dt}\begin{pmatrix} \delta A_1 \\ \delta A_2 \\ \delta A_3 \end{pmatrix} = -2A_H^2 \begin{pmatrix} 1 & G(\frac{\pi}{3}) & G(\frac{\pi}{3}) \\ G(\frac{\pi}{3}) & 1 & G(\frac{\pi}{3}) \\ G(\frac{\pi}{3}) & G(\frac{\pi}{3}) & 1 \end{pmatrix} \begin{pmatrix} \delta A_1 \\ \delta A_2 \\ \delta A_3 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = -2A_H^2(1-G(\frac{\pi}{3}))$ and $\lambda_{2,3} = -2A_H^2(1+2G(\frac{\pi}{3}))<0$.
The hex. state is stable when $\lambda_1<0$ (or $G(\frac{\pi}{3})<1$), unstable otherwise.

For the stripe solution $A_1=1$, $A_2=A_3=0$ we get

$$\frac{d}{dt}\begin{pmatrix}\delta A_1\\ \delta A_2\\ \delta A_3\end{pmatrix} = \begin{pmatrix}-2 & 0 & 0\\ 0 & 1-G(\frac{\pi}{3}) & 0\\ 0 & 0 & 1-G(\frac{\pi}{3})\end{pmatrix}\begin{pmatrix}\delta A_1\\ \delta A_2\\ \delta A_3\end{pmatrix}$$

The eigenvalues are $\lambda_1=-2<0$, $\lambda_{2,3}=1-G(\frac{\pi}{3})$, so the stripe state is stable when $\lambda_2<0$ (or $G(\frac{\pi}{3})>1$) and unstable otherwise.